

# THE KSBA COMPACTIFICATION OF THE MODULI SPACE OF $D_{1,6}$ -POLARIZED ENRIQUES SURFACES

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**ABSTRACT.** We describe the moduli compactification by stable pairs (also known as KSBA compactification) of a 4-dimensional family of Enriques surfaces, which arise as the  $\mathbb{Z}_2^2$ -covers of the blow up of  $\mathbb{P}^2$  at three general points branched along a configuration of three pairs of lines. The chosen divisor is an appropriate multiple of the ramification locus. We study the degenerations parametrized by the boundary and its stratification. We relate this compactification to the Baily-Borel compactification of the same family of Enriques surfaces. Part of the boundary of this stable pairs compactification has a toroidal behavior, another part is isomorphic to the Baily-Borel compactification, and what remains is a mixture of these two. To conclude, we construct an explicit Looijenga's semitoric compactification whose boundary strata are in bijection with the boundary strata of the KSBA compactification considered.

## 1. INTRODUCTION

In the study of moduli spaces, it is a problem to provide compactifications which are functorial and with meaningful geometric and combinatorial properties. A leading example in this sense is the Deligne-Mumford and Knudsen compactification of the moduli space of smooth  $n$ -pointed curves. Another successful example is Alexeev's compactification of the moduli space of principally polarized abelian varieties, which extended previous work of Mumford, Namikawa, and Nakamura (see [M72, N, AN, A02]). Compactifications of the moduli space of polarized K3 surfaces also received a lot of attention (see [Sc, Sh80, Sh81, L16, AT]). This motivates us to consider Enriques surfaces, together with the compactification method provided by the theory of stable pairs (see [KSB, A94, A96a, A96b] and [Fu, Ko09, Ko13a, KP, HMX] for recent developments).

In this paper we focus on the  $D_{1,6}$ -polarized Enriques surfaces, which were considered in [O].  $D_{1,6}$  denotes the sublattice of  $\mathbb{Z} \oplus \mathbb{Z}^6(-1)$  of vectors of even square. A  $D_{1,6}$ -polarization on an Enriques surface  $S$  is a primitive embedding of  $D_{1,6}$  into  $\text{Pic}(S)$  with some additional requirements (see Definition 3.2). More geometrically, the  $\mathbb{Z}_2^2$ -cover of the blow up of  $\mathbb{P}^2$  at three general points branched along a configuration of three pairs of lines is an Enriques surface with two choices of  $D_{1,6}$ -polarization (see Proposition 3.8). Our choice of divisor consists of the ramification divisor multiplied by a rational number  $\epsilon$ ,  $0 < \epsilon \ll 1$ . Let  $\overline{M}'_{D_{1,6}}$  be the normalization of the closure in the coarse moduli space of stable surface pairs of the locus of points parametrizing these Enriques surfaces with our choice of divisor (the precise definition can be found in Section 3.5).

**Theorem 1.1.** *The boundary of  $\overline{M}'_{D_{1,6}}$  has two divisorial components and one of codimension 3. For a full description of the stratification of the boundary and the degenerations parametrized by it, see Theorem 6.7.*

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Let us explain the techniques used to prove this result. Denote by  $\Delta$  the toric boundary of  $(\mathbb{P}^1)^3$  and let  $B \subset (\mathbb{P}^1)^3$  be a generic effective divisor of class  $(1, 1, 1)$ . Then the  $\mathbb{Z}_2^2$ -cover of  $B$  branched along  $\Delta|_B$  is a  $D_{1,6}$ -polarized Enriques surface (see Proposition 3.14), and we can reduce our problem to considering the pairs  $(B, (\frac{1+\epsilon}{2})\Delta|_B)$  (this is explained in Remark 3.11). The advantage of this setting is that if  $Q$  denotes the unit cube, then  $((\mathbb{P}^1)^3, B)$  is a stable toric pair of type  $\leq Q$  (see Definition 2.4), and therefore we can take advantage of the moduli theory for such pairs, which is developed in [A02]. Observe that given the stable toric pair  $((\mathbb{P}^1)^3, B)$ , then  $((\mathbb{P}^1)^3, \Delta + \epsilon B)$  is a stable pair (see [A06, Lemma 4.4]), which is not yet what we want. Let  $\overline{M}_Q^\nu$  be the normalization of the projective coarse moduli space parametrizing  $((\mathbb{P}^1)^3, B)$  and its degenerations. The next result summarizes Theorems 6.4 and 6.5.

**Theorem 1.2.** *If  $\text{Sym}(Q)$  denotes the symmetry group of  $Q$ , then  $\overline{M}_Q^\nu / \text{Sym}(Q) \cong \overline{M}_{D_{1,6}}^\nu$ . On a dense open subset, the isomorphism maps the  $\text{Sym}(Q)$ -class of a stable toric pair  $(X, B)$  to  $(B, (\frac{1+\epsilon}{2})\Delta|_B)$ , where  $\Delta$  denotes the toric boundary of  $X$ .*

One of the main problems in the proof of Theorem 1.2 is to compute the image of  $(X, B)$  when  $(B, (\frac{1+\epsilon}{2})\Delta|_B)$  is unstable, which happens if and only if the polyhedral subdivision of  $(Q, Q \cap \mathbb{Z}^3)$  associated to  $(X, B)$  has a *corner cut* (see Definition 4.1). This aspect is analyzed in Section 4, where we describe a modification of  $(X, B)$ , which we denote by  $(X^\bullet, B^\bullet)$ , such that  $(B^\bullet, (\frac{1+\epsilon}{2})\Delta^\bullet|_{B^\bullet})$  is a stable pair. Moreover, this modification can be performed in a one-parameter family. At this point, given the isomorphism of Theorem 1.2, we can explicitly describe  $\overline{M}_{D_{1,6}}^\nu$  because the moduli space  $\overline{M}_Q^\nu$  is the projective toric variety associated to the secondary polytope of  $Q$ , and it is stratified according to the (regular) polyhedral subdivisions of  $Q$ .

Consider the Baily-Borel compactification  $\overline{\mathcal{D}}/\overline{\Gamma}^{BB}$ , where  $\mathcal{D}$  is the period domain parametrizing  $D_{1,6}$ -polarized Enriques surfaces. This compactification was studied in [O], and its boundary has two rational 1-cusps and three 0-cusps. The following theorem is proved in Section 7.

**Theorem 1.3.** *There exists a birational morphism  $\overline{M}_{D_{1,6}}^\nu \rightarrow \overline{\mathcal{D}}/\overline{\Gamma}^{BB}$  extending the period map to the boundary of  $\overline{M}_{D_{1,6}}^\nu$  (see Theorem 7.8).*

In our case, proving that the period map extends is nontrivial because the boundary of  $\overline{M}_{D_{1,6}}^\nu$  is not divisorial, and therefore we cannot make use of the standard extension criterion [BJ, Corollary III.22.19]. Also in this case, we obtain the desired extension using one-parameter families. To compute the limit of a one-parameter family in the Baily-Borel compactification, it is crucial to understand the Kulikov type of a KSBA degeneration of K3 surface pairs which are the double covers of our Enriques surface pairs. This is studied in Section 7.1.

Label the 0-cusps of  $\overline{\mathcal{D}}/\overline{\Gamma}^{BB}$  even, odd of type 1 and odd of type 2 (see Figure 7). We observe that the boundary of  $\overline{M}_{D_{1,6}}^\nu$  has a toroidal behavior in a neighborhood of the preimage of the even 0-cusp, and is isomorphic to the Baily-Borel compactification in a neighborhood of the preimage of the odd 0-cusp of type 2. Above the odd 0-cusp of type 1 the behavior of  $\overline{M}_{D_{1,6}}^\nu$  is neither toroidal nor Baily-Borel. These considerations make us consider Looijenga's semitoric compactifications (see [Lo]), which generalize the Baily-Borel and toroidal compactifications in the case of type IV Hermitian symmetric domains. A semitoric compactification  $\overline{\mathcal{D}}/\overline{\Gamma}^\Sigma$  depends on a choice of combinatorial data  $\Sigma$  called an admissible decomposition of the conical locus of  $\mathcal{D}$  (see Definition 8.3). To construct  $\Sigma$  in our case, for each 0-cusp we consider the associated hyperbolic lattice, and we compute a fundamental domain for the discrete reflection group generated by the reflections with respect to

the  $(-1)$ -vectors. The computation at the odd 0-cusp of type 1 in Section 8.2.2 is noteworthy: Vinberg's algorithm produces in one step an infinite Coxeter diagram (this also happens in [Con]). We obtain the following result.

**Theorem 1.4.** *The admissible decomposition  $\Sigma$  described in Definition 8.4 produces a semitoric compactification  $\overline{\mathcal{D}}/\Gamma^\Sigma$  birational to  $\overline{M}'_{D_{1,6}}$  and whose boundary strata are in bijection with the boundary strata of  $\overline{M}'_{D_{1,6}}$ . This bijection preserves the dimensions of the strata and the intersections between them (see Theorem 8.12).*

Motivated by these observations, we make the following conjecture, which will be object of further investigation.

**Conjecture 1.5.** *The compactifications  $\overline{M}'_{D_{1,6}}$  and  $\overline{\mathcal{D}}/\Gamma^\Sigma$  are isomorphic.*

The double nature, namely toroidal and Baily-Borel, of the boundary of the moduli space  $\overline{M}'_{D_{1,6}}$  produced by the theory of stable pairs is remarkable: it illustrates the behavior one should expect in general when considering compactifications by stable pairs (see also [AT, Remark 1.5]).

After reducing the problem to considering lines on the blow up of  $\mathbb{P}^2$  at three general points, it is natural to ask whether or not the techniques in [A08, A15, AP09, Hu] apply in our case. The answer to this is no, and the reason is we do not allow any  $(-1)$ -curve to be part of our chosen divisor, making our stable pairs not compatible with only considering lines in  $\mathbb{P}^2$  (see Remark 3.12). More connections and differences with the cited works are discussed in Remarks 3.5 and 9.3. A compact moduli space for del Pezzo surfaces using stable pairs was constructed in [HKT], but in our case we have a different choice of the divisor. Our work is also related to [Hac] in the case of sextic plane curves, but here we consider the blow up  $\mathbb{P}^2$ . Some of the techniques we use in this paper are closely related to [AT] (for instance the use of stable toric pairs and polyhedral subdivisions).

The reason why we focused on this particular type of Enriques surface is the following.  $D_{1,6}$ -polarized Enriques surfaces come with a degree 6 polarization given by half the ramification locus of the  $\mathbb{Z}_2^2$ -cover (this is the polarization which gives the classical construction of Enriques). The techniques used in this paper can be generalized to describe a compactification by stable pairs of the entire 10-dimensional family of polarized Enriques surfaces of degree 6. This will be object of future work.

In Section 2 we recall the needed background about stable toric pairs and stable pairs. This theory is immediately applied in Section 3, where we define the  $D_{1,6}$ -polarized Enriques surfaces and the surface pairs we want to study. Here we also define our moduli spaces of interest:  $\overline{M}_Q$  and  $\overline{M}_{D_{1,6}}$ . Section 4 is devoted to the definition of  $(X^\bullet, B^\bullet)$  for a stable toric pair  $(X, B)$  of type  $\leq Q$ . In Section 5 we prove that  $(B^\bullet, (\frac{1+\epsilon}{2}) \Delta^\bullet|_{B^\bullet})$  is stable for every stable toric pair  $(X, B)$  of type  $\leq Q$ . In Section 6 we first construct a morphism  $\overline{M}'_Q \rightarrow \overline{M}_{D_{1,6}}$ , and then we prove the isomorphism  $\overline{M}'_Q/\text{Sym}(Q) \cong \overline{M}'_{D_{1,6}}$ , which allows us to describe the stable pairs parametrized by the boundary of  $\overline{M}'_{D_{1,6}}$  and its stratification. In Section 7 we relate  $\overline{M}'_{D_{1,6}}$  with the Baily-Borel compactification  $\overline{\mathcal{D}}/\Gamma^{BB}$ . Finally, the connection with Looijenga's semitoric compactifications is studied in Section 8.

We work over an arbitrary algebraically closed field  $\mathbb{k}$  of characteristic zero, exception made for Sections 7 and 8 where we work over  $\mathbb{C}$ .

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## 2. PRELIMINARIES: MODULI OF PAIRS

In this section we review some basic definitions and results in the theory of moduli of stable toric pairs and stable pairs. Our main references are [A02, A06, A15]. For the terminology we use from convex geometry (especially for the definition of the secondary polytope) we refer to [GKZ].

**2.1. Stable toric pairs.** For us, a *variety* is a connected and reduced scheme of finite type over an extension  $\mathbb{K}$  of  $\mathbb{k}$ , which is our fixed base field of characteristic zero (in particular, a variety need not be irreducible).  $T$  denotes a fixed (split) torus over  $\mathbb{K}$  and let  $M$  be its character lattice.  $M_{\mathbb{R}}$  denotes the tensor product  $M \otimes_{\mathbb{Z}} \mathbb{R}$ . We want to remark that the theory of stable toric pairs we are about to review works in any characteristic.

**Definition 2.1.** Let  $X$  be a variety with a  $T$ -action. We say that  $X$  is a *stable toric variety* if  $X$  is seminormal and its irreducible components are toric varieties under the  $T$ -action. The *toric boundary* of a stable toric variety is defined to be the sum of the boundary divisors of each irreducible component which are not in common with other irreducible components. If  $X$  is projective and  $\mathcal{L}$  is an ample and  $T$ -linearized invertible sheaf on  $X$ , we say that the pair  $(X, \mathcal{L})$  is a *polarized stable toric variety*.

**Remark 2.2.** Assume we have a polarized stable toric variety  $(X, \mathcal{L})$ . Then we can associate to each irreducible component  $X_i$  of  $X$  a lattice polytope  $P_i$ . These polytopes can be glued to one another in the same way as  $X$  is the union of its irreducible components. This results into a topological space  $\cup_i P_i$  which is called the *topological type* of  $X$ . The topological type comes together with a finite map  $\cup_i P_i \rightarrow M_{\mathbb{R}}$ , called the *reference map*, which embeds each  $P_i$  as a lattice

polytope in  $M_{\mathbb{R}}$ . The set of the faces of all the polytopes  $P_i$ , together with the identifications coming from the gluing, forms what is called a *complex of polytopes*. Up to isomorphism, there is a 1-to-1 correspondence between polarized stable toric varieties (for a fixed torus  $T$ ) and the following data:

- (a) A complex of polytopes  $\mathcal{P}$ ;
- (b) A reference map  $\cup_{P \in \mathcal{P}} P \rightarrow M_{\mathbb{R}}$ ;
- (c) An element of the cohomology group we are about to describe. For each  $P \in \mathcal{P}$ , let  $C_P$  be the saturated sublattice of  $\mathbb{Z} \oplus M$  generated by  $(1, P)$ , and let  $T_P$  be the torus  $\text{Hom}(C_P, \mathbb{K}^*)$ . The tori  $T_P$  for  $P \in \mathcal{P}$  define a sheaf  $\underline{T}$  on the poset  $\mathcal{P}$  with the order topology. So we want to consider an element of  $H^1(\mathcal{P}, \underline{T})$ , which describes the way the irreducible components of  $X$  are glued together.

For more details about this, see [A06, Section 4.3] or [A02, Theorem 1.2.6].

**Definition 2.3.** Let  $(X, \mathcal{L})$  be a polarized stable toric variety and let  $Q \subset M_{\mathbb{R}}$  be a lattice polytope. We say that  $X$  has *type*  $\leq Q$  if the complex of marked polytopes  $\mathcal{P}$  associated to  $X$  is a polyhedral subdivision of the marked polytope  $(Q, Q \cap M)$ . Observe that in this case the reference map is implicitly given by the inclusion of  $Q$  in  $M_{\mathbb{R}}$ . Furthermore, the toric boundary of  $X$  is given by the sum of the divisors corresponding to the facets in  $\mathcal{P}$  contained in the boundary of  $Q$ .

**Definition 2.4.** A *stable toric pair* is the datum of a polarized stable toric variety  $(X, \mathcal{L})$  together with an effective Cartier divisor  $B$  which is the divisor of zeros of a global section of  $\mathcal{L}$ . Also, we require that  $B$  does not contain any torus fixed point (or equivalently any  $T$ -orbit). We denote a stable toric pair simply by  $(X, B)$  because it is understood that  $\mathcal{L} \cong \mathcal{O}_X(B)$ . Two stable toric pairs  $(X, B)$  and  $(X', B')$  are isomorphic if there exists an isomorphism  $f: X \rightarrow X'$  that preserves the  $T$ -action and such that  $f^*B' = B$ . We say that a stable toric pair has *type*  $\leq Q$  if the corresponding polarized stable toric variety has *type*  $\leq Q$ .

**Remark 2.5.** Let  $(X, B)$  be a stable toric pair and let  $\mathcal{P}$  be the complex of polytopes corresponding to the polarized stable toric variety  $(X, \mathcal{O}_X(B))$ . If  $X_i$  is any irreducible component of  $X$ , then the restriction  $B|_{X_i}$  can be described combinatorially as follows (see [A02, Theorem 1.2.7] in combination with [A02, Lemma 2.2.7, part 2]). Consider the marking on the corresponding lattice polytope  $P_i$  given by  $P_i \cap M$ . Observe that these lattice points in  $P_i$  correspond to monomials. Now,  $B|_{X_i}$  is determined (not uniquely) by a function  $f: P_i \cap M \rightarrow \mathbb{K}^*$ , which assigns to each monomial a corresponding coefficient (which we want to be nonzero because  $B$  does not contain any torus fixed point).

**2.2. The stack of stable toric pairs.** By  $\mathbf{Sch}_{\mathbb{K}}$  we denote the category of locally noetherian schemes over our fixed base field  $\mathbb{K}$ .

**Definition 2.6.** Given  $S \in \text{Ob}(\mathbf{Sch}_{\mathbb{K}})$ , a *family of stable toric pairs*  $(\mathfrak{X}, \mathfrak{B})/S$  is the datum of a proper and flat morphism of schemes  $\pi: \mathfrak{X} \rightarrow S$ , a compatible  $T_S = T \times S$  action on  $\mathfrak{X}$ , and an effective Cartier divisor  $\mathfrak{B} \subset \mathfrak{X}$  such that  $\pi|_{\mathfrak{B}}$  is flat and the fiber  $(\mathfrak{X}_s, \mathfrak{B}_s)$  over every geometric point  $s \rightarrow S$  is a stable toric pair with the action induced by  $T_S$  (this makes sense because in Section 2.1 we defined stable toric pairs over an arbitrary field extension of  $\mathbb{K}$ ). Two families of stable toric pairs over the same base are *isomorphic* if there exists an isomorphism of pairs over  $S$  preserving the torus action. Given a lattice polytope  $Q$ , we say that a family of stable toric pairs has *type*  $\leq Q$  if every geometric fiber has *type*  $\leq Q$ .

**Remark 2.7.** Let  $(\mathfrak{X}, \mathfrak{B})/S$  be a family of stable toric pairs. Denote by  $\pi$  the morphism  $\mathfrak{X} \rightarrow S$  and let  $\mathcal{L} = \mathcal{O}_{\mathfrak{X}}(\mathfrak{B})$ . Following [A02, Proof of Lemma 2.10.1], for  $d \geq 0$  the sheaves  $\pi_* \mathcal{L}^{\otimes d}$  are locally free, and the torus action gives a decomposition  $\pi_* \mathcal{L}^{\otimes d} = \bigoplus_{m \in M} R_{(d,m)}$  into sheaves that are also locally free. This results into a locally free  $(\mathbb{Z} \oplus M)$ -graded  $\mathcal{O}_S$ -algebra  $R = \bigoplus_{(d,m) \in \mathbb{Z} \oplus M} R_{(d,m)}$  together with a  $\mathbb{Z}$ -degree 1 section  $\theta$  of  $R$  corresponding to  $\mathfrak{B}$ . Conversely a pair  $(R, \theta)$ , where  $R$  is a locally free graded  $\mathcal{O}_S$ -algebra and  $\theta$  a degree 1 section, uniquely determines a family of stable toric pairs up to isomorphism (see [A02, Proof of Theorem 2.10.8] or [A06, Section 4]).

**Remark 2.8.** After defining families of stable toric pairs we automatically have a notion of stack over  $\mathbb{k}$ . Given a lattice polytope  $Q$ , denote by  $\mathcal{M}_Q$  the category of families of stable toric pairs of type  $\leq Q$ , where a morphism  $f = (f_t, f_b): (\mathfrak{X}', \mathfrak{B}')/S' \rightarrow (\mathfrak{X}, \mathfrak{B})/S$  is a pullback diagram

$$\begin{array}{ccc} (\mathfrak{X}', \mathfrak{B}') & \xrightarrow{f_t} & (\mathfrak{X}, \mathfrak{B}) \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f_b} & S. \end{array}$$

Let  $\mathcal{M}_Q \rightarrow \mathbf{Sch}_{\mathbb{k}}$  be the functor sending  $(\mathfrak{X}, \mathfrak{B})/S$  to  $S$  and a morphism  $f$  to  $f_b$ . Then  $\mathcal{M}_Q$  is a stack over  $\mathbb{k}$  parametrizing families of stable toric pairs of type  $\leq Q$ .

The main result about the stack  $\mathcal{M}_Q$  is the following theorem due to Alexeev.

**Theorem 2.9** ([A02, Theorem 1.2.15]). *Let  $Q$  be a lattice polytope and let  $\mathcal{M}_Q$  be the stack of stable toric pairs of type  $\leq Q$ . Then the following hold:*

- (i)  $\mathcal{M}_Q$  is a proper quotient stack (and therefore an Artin stack) with finite stabilizers<sup>1</sup>;
- (ii) It has a coarse moduli space  $\overline{M}_Q$  which is a projective scheme;
- (iii)  $\overline{M}_Q$  is naturally stratified, and every stratum corresponds in a 1-to-1 way to a polyhedral subdivision of  $(Q, Q \cap M)$ ;
- (iv) The normalization of the main irreducible component of  $(\overline{M}_Q)_{\text{red}}$  is isomorphic to the toric variety associated to the secondary polytope  $\Sigma(Q \cap M)$ .

### 2.3. Stable pairs.

**Definition 2.10.** Let  $X$  be a variety and let  $B = \sum_i b_i B_i$  be a divisor on  $X$  where, for all  $i$ ,  $b_i \in (0, 1] \cap \mathbb{Q}$  and  $B_i$  is a prime divisor. Then the pair  $(X, B)$  is *semi-log canonical* if the following conditions are satisfied:

- (i)  $X$  is  $S_2$  and every codimension 1 point is regular or a double crossing singularity (varieties satisfying these properties are also called *demi-normal*);
- (ii) if  $\nu: X^\nu \rightarrow X$  is the normalization with conductors  $D \subset X$  and  $D^\nu \subset X^\nu$  (see [Ko13b, Section 5.1]), then the support of  $B$  does not contain any irreducible component of  $D$ ;
- (iii)  $K_X + B$  is  $\mathbb{Q}$ -Cartier;
- (iv) the pair  $(X^\nu, D^\nu + \nu_*^{-1}B)$  is *log canonical* (see [KM, Definition 2.34]).

**Definition 2.11.** A pair  $(X, B)$  is *stable* if the following conditions are satisfied:

- (i) *on singularities:*  $(X, B)$  is a semi-log canonical pair;
- (ii) *numerical:*  $K_X + B$  is ample.

<sup>1</sup>Since we are working in characteristic 0, we have that  $\mathcal{M}_Q$  is actually a Deligne-Mumford stack.

## 2.4. The Viehweg's moduli stack.

**Definition 2.12.** The Viehweg's moduli stack  $\overline{\mathcal{M}}_{d,N,C,\underline{b}}$  is defined as follows. Let us fix constants  $d, N \in \mathbb{Z}_{>0}$ ,  $C \in \mathbb{Q}_{>0}$ , and  $\underline{b} = (b_1, \dots, b_n)$  with  $b_i \in (0, 1] \cap \mathbb{Q}$  and  $Nb_i \in \mathbb{Z}$  for all  $i = 1, \dots, n$ . For any scheme  $S$  over  $\mathbb{k}$ ,  $\overline{\mathcal{M}}_{d,N,C,\underline{b}}(S)$  is the set of proper flat families  $\mathfrak{X} \rightarrow S$  together with a divisor  $\mathfrak{B} = \sum_i b_i \mathfrak{B}_i$  satisfying the following properties:

- For all  $i = 1, \dots, n$ ,  $\mathfrak{B}_i$  is a codimension 1 closed subscheme which is flat over  $S$ ;
- Every geometric fiber  $(X, B)$  is a stable pair of dimension  $d$  with  $(K_X + B)^d = C$ ;
- There exists an invertible sheaf  $\mathcal{L}$  on  $\mathfrak{X}$  such that for every geometric fiber  $(X, B)$  one has  $\mathcal{L}|_X \cong \mathcal{O}_X(N(K_X + B))$ .

**Remark 2.13.** Whether or not the stack  $\overline{\mathcal{M}}_{d,N,C,\underline{b}}$  is coarsely represented by a projective scheme is a subtle matter. In our case (see Section 3.5) it is. More details can be found in [A15, Vie].

## 3. $D_{1,6}$ -POLARIZED ENRIQUES SURFACES AND THE CHOICE OF THE DIVISOR

**3.1. The  $D_{1,6}$  polarization.** For us, a surface is a 2-dimensional projective variety. An *Enriques surface*  $Y$  is a smooth irreducible surface with  $2K_Y \sim 0$  and  $h^0(Y, \omega_Y) = h^1(Y, \mathcal{O}_Y)$  (these properties are enough to imply that  $S$  is minimal with Kodaira dimension 0 and  $h^0(Y, \omega_Y), h^1(Y, \mathcal{O}_Y) = 0$ ).

**Definition 3.1.** Let  $\mathbb{Z}^{1,6}$  denote the unique odd unimodular lattice of signature  $(1, 6)$ . Then the  $D_{1,6}$  lattice is the index 2 sublattice of  $\mathbb{Z}^{1,6}$  containing vectors with even square (or equivalently, we can require the sum of the coordinates to be even). If  $e_0, e_1, \dots, e_6$  is the canonical basis of  $\mathbb{Z}^{1,6}$ , then we distinguish the following vectors in  $D_{1,6}$ :  $2e_0, e_1 \pm e_2, e_3 \pm e_4, e_5 \pm e_6$ .

**Definition 3.2.** A  $D_{1,6}$ -polarized Enriques surface is an Enriques surface whose Picard group contains a primitively embedded copy of  $D_{1,6}$  such that:

- (a) The distinguished vector  $2e_0$  corresponds to a nef divisor class  $H$  (observe that  $H$  is also big because  $H^2 = (2e_0)^2 = 4$ );
- (b) The distinguished vectors  $e_1 \pm e_2, e_3 \pm e_4, e_5 \pm e_6$  correspond to six irreducible curves  $R_1^\pm, R_2^\pm, R_3^\pm$  respectively (these curves are  $(-2)$ -curves, i.e. isomorphic to  $\mathbb{P}^1$  and with self-intersection  $-2$ ).

The next proposition is well known from [O, Section 3]. However, given its importance for us, we briefly sketch its proof.

**Proposition 3.3.** *Let  $S$  be a  $D_{1,6}$ -polarized Enriques surface. Then the linear system  $|H - R_i^+ - R_i^-|$  is a genus 1 pencil,  $i = 1, 2, 3$ .*

*Proof.* The linear system  $|H|$  is base point free and 2-dimensional (see [O, Proposition 3.1]). Moreover,  $|H|$  contracts the curves  $R_i^\pm$ ,  $i = 1, 2, 3$ , because  $H \cdot R_i^\pm = 0$ . On the other hand,  $|H - R_i^+ - R_i^-|$  is a proper linear subsystem of  $|H|$  (for instance, it does not contract  $R_i^\pm$  and contracts  $R_j^\pm$  for  $j \neq i$ ). Therefore,  $|H - R_i^+ - R_i^-|$  is a pencil and a curve  $C$  in this linear system has (arithmetic) genus 1 because  $C^2 = (H - R_i^+ - R_i^-)^2 = 0$  (now use [B, I.15 The genus formula]).  $\square$

**Observation 3.4.** Every genus 1 fibration on an Enriques surface has exactly two double fibers which are called *half-fibers* or *half-pencils* (see [BHPV, Chapter VIII, Section 17]). Therefore, for a  $D_{1,6}$ -polarized Enriques surface  $S$ , denote by  $E_i$  and  $E'_i$  the two half-fibers in the genus 1 pencil  $|H - R_i^+ - R_i^-|$ ,  $i = 1, 2, 3$ . Then the divisor  $\sum_{i=1}^3 (E_i + E'_i)$  is canonically associated to the  $D_{1,6}$ -polarization, and its intrinsic nature leads us to consider moduli of pairs  $(S, \epsilon \cdot \sum_{i=1}^3 (E_i + E'_i))$  for

$0 < \epsilon \ll 1$ ,  $\epsilon \in \mathbb{Q}$ . Observe also that the divisor  $\sum_{i=1}^3 E_i$  is a degree 6 polarization which gives the classical construction of Enriques: it realizes  $S$  as the normalization of a degree 6 surface in  $\mathbb{P}^3$  with only double crossing singularities along the edges of the coordinate tetrahedron.

**Remark 3.5.** If  $S$  is a  $D_{1,6}$ -polarized Enriques surface, then the divisor  $C = H + \sum_{i=1}^3 (R_i^+ + R_i^-)$  is divisible by 2 in  $\text{Pic}(S)$ . The  $\mathbb{Z}_2$ -cover of  $S$  branched along  $C$  has six  $(-1)$ -curves. Blowing down these curves we obtain a Campedelli surface with (topological) fundamental group  $\mathbb{Z}_2^3$  (these were considered in [AP09]). Conversely, such a Campedelli surface  $X$  can be realized as the  $\mathbb{Z}_2^3$ -cover of  $\mathbb{P}^2$  branched along seven lines. The minimal desingularization of the quotient of  $X$  by the involution fixing pointwise the preimage of one of these lines is a  $D_{1,6}$ -polarized Enriques surface.

**Remark 3.6.** By [Cos, Theorem 3] we know that every Enriques surface contains three genus 1 pencils  $|2E_1|, |2E_2|, |2E_3|$  such that  $E_1 \cdot E_2 = E_2 \cdot E_3 = E_3 \cdot E_1 = 1$ . Moreover, observe that Enriques surfaces with a  $D_{1,6}$ -polarization are a specialization of the family of Enriques surfaces considered in [Ve].

### 3.2. $D_{1,6}$ -polarized Enriques surfaces as $\mathbb{Z}_2^2$ -covers.

**Notation 3.7.** Let  $\text{Bl}_3 \mathbb{P}^2$  be the blow up of  $\mathbb{P}^2$  at  $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ .  $\text{Bl}_3 \mathbb{P}^2$  comes with three genus 0 pencils  $\pi_1, \pi_2, \pi_3 : \text{Bl}_3 \mathbb{P}^2 \rightarrow \mathbb{P}^1$ , and denote by  $\ell_i, \ell'_i$  two distinct irreducible elements in the  $i$ -th pencil,  $i = 1, 2, 3$ .

**Proposition 3.8.** *Let  $S$  be a  $D_{1,6}$ -polarized Enriques surface. Then there exists a divisor  $\sum_{i=1}^3 (\ell_i + \ell'_i)$  on  $\text{Bl}_3 \mathbb{P}^2$  (see Notation 3.7) without triple intersection points and a morphism  $\pi : S \rightarrow \text{Bl}_3 \mathbb{P}^2$  such that*

- (i)  $\pi$  is the  $\mathbb{Z}_2^2$ -cover of  $\text{Bl}_3 \mathbb{P}^2$  ramified at  $\sum_{i=1}^3 (E_i + E'_i)$  and branched along  $\sum_{i=1}^3 (\ell_i + \ell'_i)$ ;
- (ii) The linear system  $|2E_i|$  is the pullback of  $|\ell_i|$ ,  $i = 1, 2, 3$ .

*Conversely, given any divisor  $\sum_{i=1}^3 (\ell_i + \ell'_i)$  on  $\text{Bl}_3 \mathbb{P}^2$  without triple intersection points, the  $\mathbb{Z}_2^2$ -cover of  $\text{Bl}_3 \mathbb{P}^2$  branched along  $\sum_{i=1}^3 (\ell_i + \ell'_i)$  is an Enriques surface with exactly two choices of  $D_{1,6}$ -polarization:  $H$  is the pullback of a generic line in  $\text{Bl}_3 \mathbb{P}^2$ , and the six lines  $R_i^\pm$ ,  $i = 1, 2, 3$ , are the preimages of the exceptional divisors in  $\text{Bl}_3 \mathbb{P}^2$ , or the other three  $(-1)$ -curves. These two choices are exchanged by the Cremona involution of  $\text{Bl}_3 \mathbb{P}^2$  (see Observation 3.9).*

*Proof.* (i) and (ii) are discussed in [O, Section 3]. The divisor  $\sum_{i=1}^3 (\ell_i + \ell'_i)$  cannot have triple intersection points otherwise  $S$  would be singular (see [Pa, Proposition 3.1]).

For the inverse construction, we use the general theory of abelian covers developed in [Pa]. Let  $\mathbb{Z}_2^2 = \{e, a, b, c\}$  where  $e$  is the identity element, and let  $\{\chi_0, \chi_1, \chi_2, \chi_3\}$  be the characters of  $\mathbb{Z}_2^2$  with  $\chi_0 = 1$  and  $\chi_1(b) = \chi_1(c) = \chi_2(a) = \chi_2(c) = \chi_3(a) = \chi_3(b) = -1$ . Define  $D_a = \ell_2 + \ell'_2 + \ell_3 + \ell'_3$ ,  $D_b = \ell_1 + \ell'_1 + \ell_3 + \ell'_3$ ,  $D_c = \ell_1 + \ell'_1 + \ell_2 + \ell'_2$ . Consider the building data (see [Pa, Definition 2.1]) consisting of the divisors  $D_a, D_b, D_c$  and the line bundles  $L_{\chi_1}, L_{\chi_2}, L_{\chi_3}$  satisfying

$$2L_{\chi_1} = D_b + D_c, \quad 2L_{\chi_2} = D_a + D_c, \quad 2L_{\chi_3} = D_a + D_b.$$

This building data determines a  $\mathbb{Z}_2^2$ -cover  $\pi : S \rightarrow \text{Bl}_3 \mathbb{P}^2$  branched along  $\sum_{i=1}^3 (\ell_i + \ell'_i)$  which is unique up to isomorphism of  $\mathbb{Z}_2^2$ -covers (see [Pa, Theorem 2.1]). By [Pa, Proposition 3.1] we have that  $S$  is smooth, and using [Pa, Proposition 4.2, formula (4.8)] one can easily compute that  $\chi(\mathcal{O}_S) = 1$ , which tells us that  $h^0(S, \omega_S) = h^1(S, \mathcal{O}_S)$ . If  $R$  denotes the ramification divisor of the cover, then  $K_S \sim \pi^*(K_{\text{Bl}_3 \mathbb{P}^2}) + R$  and  $2R \sim \pi^*(\sum_{i=1}^3 (\ell_i + \ell'_i))$  imply that  $2K_S \sim 0$ , hence  $S$  is an Enriques surface.  $\square$



**Observation 3.9.** In the statement of Proposition 3.8, the ambiguity for the choice of  $D_{1,6}$ -polarization on the  $\mathbb{Z}_2^2$ -cover of  $\text{Bl}_3 \mathbb{P}^2$  branched along  $\sum_{i=1}^3 (\ell_i + \ell'_i)$  is not an issue for our purposes. This is because the Cremona involution on  $\text{Bl}_3 \mathbb{P}^2$  does not change the isomorphism class of the pair  $(\text{Bl}_3 \mathbb{P}^2, \sum_{i=1}^3 (\ell_i + \ell'_i))$ , even though it changes the line configuration.

**Lemma 3.10.** *Let  $S$  be a  $D_{1,6}$ -polarized Enriques surface and let  $\pi: S \rightarrow \text{Bl}_3 \mathbb{P}^2$  be the corresponding  $\mathbb{Z}_2^2$ -cover ramified at  $E = \sum_{i=1}^3 (E_i + E'_i)$  and branched along  $L = \sum_{i=1}^3 (\ell_i + \ell'_i)$  (see Proposition 3.8). Then*

$$K_S + \epsilon E \sim_{\mathbb{Q}} \pi^* \left( K_{\text{Bl}_3 \mathbb{P}^2} + \left( \frac{1+\epsilon}{2} \right) L \right).$$

*In particular,  $(S, \epsilon E)$  is stable if and only if  $(\text{Bl}_3 \mathbb{P}^2, (\frac{1+\epsilon}{2}) L)$  is stable.*

*Proof.* We have that  $K_S \sim \pi^*(K_{\text{Bl}_3 \mathbb{P}^2}) + E$  and  $2E \sim \pi^*(L)$ . This implies that  $K_S \sim_{\mathbb{Q}} \pi^*(K_{\text{Bl}_3 \mathbb{P}^2}) + \frac{1}{2}\pi^*(L)$ , and by adding  $\epsilon E \sim_{\mathbb{Q}} \frac{\epsilon}{2}\pi^*(L)$  to both sides we obtain what claimed. For the last statement about stability, we have that  $(S, \epsilon E)$  is semi-log canonical if and only if  $(\text{Bl}_3 \mathbb{P}^2, (\frac{1+\epsilon}{2}) L)$  is semi-log canonical by [AP12, Lemma 2.3], and  $K_S + \epsilon E$  is ample if and only if  $K_{\text{Bl}_3 \mathbb{P}^2} + (\frac{1+\epsilon}{2}) L$  is ample by [L, Proposition 1.2.13 and Corollary 1.2.28].  $\square$

**Remark 3.11.** With the notation introduced in Lemma 3.10, we claim that studying the degenerations of the stable pairs  $(S, \epsilon E)$  is equivalent to the study of the degenerations of the stable pairs  $(\text{Bl}_3 \mathbb{P}^2, (\frac{1+\epsilon}{2}) L)$ . To prove this, let  $K$  be the field of fractions of a DVR  $(A, \mathfrak{m})$ , where  $\mathfrak{m}$  is the maximal ideal of  $A$ . Let  $(\mathfrak{S}^\circ, \mathcal{E}^\circ)$  (resp.  $(\mathfrak{B}^\circ, \mathcal{L}^\circ)$ ) be a family of stable pairs over  $\text{Spec}(K)$  with fibers isomorphic to  $(S, \epsilon E)$  (resp.  $(\text{Bl}_3 \mathbb{P}^2, (\frac{1+\epsilon}{2}) L)$ ). Let  $\mathfrak{S}^\circ \rightarrow \mathfrak{B}^\circ$  be the  $\mathbb{Z}_2^2$ -cover ramified at  $\mathcal{E}^\circ$  and branched along  $\mathcal{L}^\circ$ . Let  $(\mathfrak{S}, \mathcal{E})$  be the completion of  $(\mathfrak{S}^\circ, \mathcal{E}^\circ)$  over  $\text{Spec}(A)$ , or a ramified base change of it, having as central fiber a stable pair (see [A06, Theorem 2.1]). Denote by  $(\mathfrak{S}_{\mathfrak{m}}, \mathcal{E}_{\mathfrak{m}})$  the central fiber of  $(\mathfrak{S}, \mathcal{E})$ . We adopt similar notations for  $(\mathfrak{B}^\circ, \mathcal{L}^\circ)$ . What we need to show is that  $(\mathfrak{S}_{\mathfrak{m}}, \mathcal{E}_{\mathfrak{m}})$  is the  $\mathbb{Z}_2^2$ -cover of  $(\mathfrak{B}_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}})$ . This is automatic if we can show that the  $\mathbb{Z}_2^2$ -action on  $\mathfrak{S}^\circ$  extends to  $\mathfrak{S}$  (in this case, the quotient of  $\mathfrak{S}$  by  $\mathbb{Z}_2^2$  is isomorphic to  $\mathfrak{B}$  by the uniqueness of the completion of  $\mathfrak{B}^\circ$  over  $\text{Spec}(A)$ , and this implies what we want). Fix any  $g \in \mathbb{Z}_2^2$  and consider its corresponding action  $\alpha_g: \mathfrak{S} \dashrightarrow \mathfrak{S}$ . Resolving the indeterminacies of  $\alpha_g$ , we obtain a morphism  $\alpha'_g: \mathfrak{S}' \rightarrow \mathfrak{S}$  where  $\mathfrak{S}'$  is obtained by blowing up  $\mathfrak{S}$ . Then  $\alpha'_g$  corresponds to a morphism  $\overline{\alpha}_g$  from the log canonical model of  $\mathfrak{S}'$  to the log canonical model of  $\mathfrak{S}$ , which are both isomorphic to  $\mathfrak{S}$ . Then  $\overline{\alpha}_g: \mathfrak{S} \rightarrow \mathfrak{S}$  is the desired extension of  $\alpha_g$ .

**Remark 3.12.** As we already commented in the Introduction, after reducing our problem to considering lines on  $\text{Bl}_3 \mathbb{P}^2$ , we cannot use the hyperplane arrangements machinery from [A08, A15, AP09]. The reason is that the divisor  $L = \sum_{i=1}^3 (\ell_i + \ell'_i)$  does not contain any  $(-1)$ -curve, and therefore  $L$  is not the pullback from  $\mathbb{P}^2$  of a line arrangement.

### 3.3. Reduction to stable toric pairs.

**Notation 3.13.** Let  $([X_0 : X_1], [Y_0 : Y_1], [Z_0 : Z_1])$  be coordinates in  $(\mathbb{P}^1)^3$ . We denote by  $\Delta$  the toric boundary  $V(X_0 X_1 Y_0 Y_1 Z_0 Z_1) \subset (\mathbb{P}^1)^3$  ( $V(I)$  denotes the closed subscheme associated to the homogeneous ideal  $I$ ).

**Proposition 3.14.** *Consider  $\text{Bl}_3 \mathbb{P}^2$  together with a divisor  $\sum_{i=1}^3 (\ell_i + \ell'_i)$  (see Notation 3.7) without triple intersection points. Then there exists  $B = V \left( \sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k \right) \subset (\mathbb{P}^1)^3$  with coefficients*

$c_{ijk} \neq 0$  such that  $(\text{Bl}_3 \mathbb{P}^2, \sum_{i=1}^3 (\ell_i + \ell'_i))$  is isomorphic to  $(B, \Delta|_B)$ . Moreover, such  $B \subset (\mathbb{P}^1)^3$  is uniquely determined up to the action of  $\mathbb{G}_m^3 \rtimes \text{Sym}(Q)$ , where  $Q$  is the unit cube.

*Proof.* Consider the three projections  $\pi_i: \text{Bl}_3 \mathbb{P}^2 \rightarrow \mathbb{P}^1$ ,  $i = 1, 2, 3$ . Let  $\ell_i = \pi_i^{-1}([a_{0i} : a_{1i}])$  and  $\ell'_i = \pi_i^{-1}([a'_{0i} : a'_{1i}])$ . The morphism  $(\pi_1, \pi_2, \pi_3): \text{Bl}_3 \mathbb{P}^2 \rightarrow (\mathbb{P}^1)^3$  is an embedding whose image is a divisor of class  $(1, 1, 1)$  (observe that the restriction of  $\Delta$  to this divisor gives the six  $(-1)$ -curves, each one with multiplicity 2). For each one of the three copies of  $\mathbb{P}^1$  (which we label with  $i = 1, 2, 3$ ) choose an automorphism  $\varphi_i$  sending  $[a_{0i} : a_{1i}], [a'_{0i} : a'_{1i}]$  to  $[1 : 0], [0 : 1]$  respectively. Let  $B$  be the image of the composition of the embedding  $(\pi_1, \pi_2, \pi_3)$  followed by the automorphism  $(\varphi_1, \varphi_2, \varphi_3)$ . Then, under this morphism,  $\text{Bl}_3 \mathbb{P}^2 \cong B$  and  $\sum_{i=1}^3 (\ell_i + \ell'_i)$  corresponds to  $\Delta|_B$ . Moreover,  $B = V\left(\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k\right)$  where the coefficients  $c_{ijk}$  are nonzero (otherwise  $\Delta|_B$  would have triple intersection points).

In the construction of  $B$  above we made some choices. There is a  $\text{Sym}(Q)$ -action which comes from the fact that we can permute the three projections  $\pi_1, \pi_2, \pi_3$  and for each  $i$  we can exchange  $[a_{0i} : a_{1i}]$  with  $[a'_{0i} : a'_{1i}]$  ( $\text{Sym}(Q)$  is isomorphic to the wreath product  $\mathbb{Z}_2 \wr S_3$ ). There is also a  $\mathbb{G}_m^3$ -action due to the fact that each  $\varphi_i$  is uniquely determined up to  $\mathbb{G}_m$ . It is easy to observe that our construction of  $B$  is invariant under the action of  $\text{Aut}(\text{Bl}_3 \mathbb{P}^2)$  on the line arrangement (see [D, Theorem 8.4.2] for the description of  $\text{Aut}(\text{Bl}_3 \mathbb{P}^2)$ ).

To conclude we need to show that any realization  $B = V\left(\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k\right) \subset (\mathbb{P}^1)^3$  with coefficients  $c_{ijk} \neq 0$  such that  $(\text{Bl}_3 \mathbb{P}^2, \sum_{i=1}^3 (\ell_i + \ell'_i)) \cong (B, \Delta|_B)$  can be obtained as we described up to  $(\mathbb{G}_m^3 \rtimes \text{Sym}(Q))$ -action. But this is true because, up to  $\mathbb{G}_m^3 \rtimes \text{Sym}(Q)$ , there is a unique way to realize  $\text{Bl}_3 \mathbb{P}^2$  in  $(\mathbb{P}^1)^3$  so that the six  $(-1)$ -curves are given by the restriction of  $\Delta$ , and this is given by  $V(X_0 Y_0 Z_0 - X_1 Y_1 Z_1)$  (we omit the proof of this).  $\square$

Summing up, starting from our pairs of interest  $(S, \epsilon \cdot \sum_{i=1}^3 (E_i + E'_i))$  defined in Observation 3.4, the considerations we made so far (Proposition 3.8, Lemma 3.10, Remark 3.11, and Proposition 3.14) allow us to consider moduli of the pairs  $(B, (\frac{1+\epsilon}{2}) \Delta|_B)$  with  $B \subset (\mathbb{P}^1)^3$  as prescribed by Proposition 3.14 (observe that  $(B, (\frac{1+\epsilon}{2}) \Delta|_B)$  is obviously stable). This approach in terms of  $B \subset (\mathbb{P}^1)^3$  is convenient because  $((\mathbb{P}^1)^3, B)$  is a stable toric pair of type  $\leq Q$  where  $Q$  is the unit cube, and Theorem 2.9 gives an explicit description of the moduli space  $\overline{M}_Q$  parametrizing  $((\mathbb{P}^1)^3, B)$  and its degenerations.

**Remark 3.15.** Let  $B \subset (\mathbb{P}^1)^3$  as in Proposition 3.14. An alternative construction of the  $\mathbb{Z}_2^2$ -cover of  $B$  branched along  $\Delta|_B$  is the following. Let  $\tilde{S} = V\left(\sum_{i,j,k=0,1} c_{ijk} X_i^2 Y_j^2 Z_k^2\right) \subset (\mathbb{P}^1)^3$ . The restriction to  $\tilde{S}$  of the map  $(\mathbb{P}^1)^3 \rightarrow (\mathbb{P}^1)^3$  given in an affine patch by  $(x, y, z) \mapsto (x^2, y^2, z^2)$  is the  $\mathbb{Z}_2^2$ -cover of  $B$  branched along  $\Delta|_B$ . To show that  $\tilde{S}$  is smooth, we first observe that  $\tilde{S}$  is smooth in the complement of the ramification locus because  $B$  is smooth. If  $\tilde{S}$  is singular at a point in the ramification locus, then one can explicitly compute using the equation of  $\tilde{S}$  that one of the irreducible components of  $\Delta$  restricts to  $B$  giving a reducible curve, which is against our assumptions. Therefore  $\tilde{S}$  is smooth, implying that  $\tilde{S}$  is a K3 surface ( $K_{\tilde{S}} \sim 0$  by adjunction formula and  $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$  can be computed using the long exact sequence in cohomology associated to  $0 \rightarrow \mathcal{I}_{\tilde{S}} \rightarrow \mathcal{O}_{(\mathbb{P}^1)^3} \rightarrow \mathcal{O}_{\tilde{S}} \rightarrow 0$ ). Observe that  $\tilde{S}$  is invariant under the involution  $\iota: (\mathbb{P}^1)^3 \rightarrow (\mathbb{P}^1)^3$  given in an affine patch by  $(x, y, z) \mapsto (-x, -y, -z)$ . Furthermore,  $\iota|_{\tilde{S}}$  has no fixed points (these

being the torus fixed points of  $(\mathbb{P}^1)^3$ , which are not contained in  $\tilde{S}$ ). This implies that  $S = \tilde{S}/\langle \iota \rangle$  is an Enriques surface (see [B, Proposition VIII.17]) which is the  $\mathbb{Z}_2^3/\langle \iota \rangle \cong \mathbb{Z}_2^2$ -cover of  $B$  branched along  $\Delta|_B$ .

**3.4. The moduli space  $\overline{M}_Q$ .** Let  $B \subset (\mathbb{P}^1)^3$  as in Proposition 3.14. Denote by  $Q$  the unit cube. Then, according to Definition 2.4,  $((\mathbb{P}^1)^3, B)$  is a stable toric pair of type  $\leq Q$ . Let  $\overline{M}_Q$  be the projective coarse moduli space parametrizing these pairs and their degenerations (see Theorem 2.9). In the next proposition we prove some first properties of  $\overline{M}_Q$ , and we can see how the geometry of  $\overline{M}_Q$  interacts with the combinatorics of the polyhedral subdivisions of  $(Q, Q \cap \mathbb{Z}^3)$ .

**Proposition 3.16.** *Let  $Q \subset \mathbb{Z}^3$  be the unit cube. Then the coarse moduli space  $\overline{M}_Q$  is irreducible and has dimension 4.*

*Proof.* The irreducibility of  $\overline{M}_Q$  follows from the fact that, if  $\mathcal{P}$  is any polyhedral subdivisions of  $(Q, Q \cap \mathbb{Z}^3)$ , then  $\mathcal{P}$  is regular and  $H^1(\mathcal{P}, \underline{\mathbb{Z}}) = \{1\}$  (see Remark 2.2). To prove the former, we know from [Sa, Corollary 2.9] that a marked polytope with a nonregular subdivision has a nonregular triangulation. But all the triangulations of  $(Q, Q \cap \mathbb{Z}^3)$  are regular (see [DL, Theorem 3.2] or [DLRS, Theorem 6.3.10]). The proof that  $H^1(\mathcal{P}, \underline{\mathbb{Z}}) = \{1\}$  can be found in Section 10.1.

For the dimension count, we know that the normalization of  $(\overline{M}_Q)_{\text{red}}$  is the projective toric variety associated to the secondary polytope  $\Sigma(Q \cap \mathbb{Z}^3)$  (see [Pf] for a complete description of this polytope), which has dimension  $\#(Q \cap \mathbb{Z}^3) - \dim(Q) - 1 = 4$  (see [GKZ, Chapter 7, Theorem 1.7]). In conclusion,  $\dim(\overline{M}_Q) = 4$ .  $\square$

**3.5. The moduli space  $\overline{M}_{D_{1,6}}$ .** With reference to Section 2.4, consider the moduli stack  $\overline{\mathcal{M}}_C = \overline{\mathcal{M}}_{d,N,C,\underline{b}}$  with  $d = 2$ ,  $\underline{b} = (b_1, b_2, b_3) = (\frac{1+\epsilon}{2}, \frac{1+\epsilon}{2}, \frac{1+\epsilon}{2})$  (because we want three pairs of divisors, and we do not distinguish divisors in the same pair) where  $0 < \epsilon \ll 1$  is a fixed rational number and  $C = 6\epsilon^2$  (given  $(B, (\frac{1+\epsilon}{2})\Delta|_B)$  as in Proposition 3.14, it is easy to compute that  $(K_B + (\frac{1+\epsilon}{2})\Delta|_B)^2 = 6\epsilon^2$ ). For a suitably chosen positive integer  $N$  depending on  $d, \underline{b}$  and  $C$  (which does not need to be specified, see [A96b, 3.13]), the Viehweg's moduli functor  $\overline{\mathcal{M}}_{6\epsilon^2}$  above is coarsely represented by a projective scheme, which we denote by  $\overline{M}_{6\epsilon^2}$ . This is true because we are working with surface pairs ( $d = 2$ ) and our coefficients are strictly greater than  $\frac{1}{2}$  (see [A15, Theorem 1.6.1, case 2]).

**Observation 3.17.** The group  $S_3$  has a natural action on the Viehweg moduli stack  $\overline{\mathcal{M}}_{6\epsilon^2}$  given by permuting the labels of the three divisors  $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$ . In particular, we have an induced  $S_3$ -action on the coarse moduli space  $\overline{M}_{6\epsilon^2}$ .

Now we want to identify inside  $\overline{M}_{6\epsilon^2}/S_3$  the Zariski closed subscheme containing a dense open subset whose points parametrize the stable pairs given by  $\text{Bl}_3 \mathbb{P}^2$  together with our three undistinguished pairs of lines of weight  $\frac{1+\epsilon}{2}$ . The next lemma allows us to do so.

**Lemma 3.18.** *The coarse moduli space  $\overline{M}_Q$  contains a dense open subset  $\mathcal{U}$  which is a fine moduli space parametrizing all the stable toric pairs  $(X, B)$  with  $X \cong (\mathbb{P}^1)^3$ .  $\mathcal{U}$  is isomorphic to the dense open subtorus of the toric variety  $\overline{M}_Q^\vee$ . The universal family on  $\mathcal{U}$  is of the form  $((\mathbb{P}^1)^3 \times \mathcal{U}, \mathfrak{S})$ .*

*Proof.* Let  $\mathcal{P}$  be the complex of polytopes given by  $Q$  and its faces. Let  $C$  be the set of vertices of  $Q$ . Following [A02, Definition 2.6.6], let  $\text{MP}^{\text{fr}}[\mathcal{P}, C](\mathbb{k})$  be the groupoid of stable toric pairs  $((\mathbb{P}^1)^3, B)$  over  $\mathbb{k}$  with the linearized line bundle  $\mathcal{O}_{(\mathbb{P}^1)^3}(B)$  in which the arrows are the isomorphisms identical on the torus  $T$ . By [A02, Lemma 2.6.7], we have that  $\text{MP}^{\text{fr}}[\mathcal{P}, C](\mathbb{k})$  is equivalent to

the quotient stack  $[\mathbb{G}_m^8/\mathbb{G}_m^4]$ , where  $\mathbb{G}_m^8$  represents the space of coefficients of the divisor  $B = V\left(\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k\right)$  on  $(\mathbb{P}^1)^3$  and  $(\lambda, \mu_1, \mu_2, \mu_3) \in \mathbb{G}_m^4$  acts on  $\mathbb{G}_m^8$  as follows:

$$(\lambda, \mu_1, \mu_2, \mu_3) \cdot (\dots, c_{ijk}, \dots) = (\dots, \lambda \mu_1^i \mu_2^j \mu_3^k c_{ijk}, \dots).$$

Observe that the quotient  $\mathbb{G}_m^8/\mathbb{G}_m^4$  exists as a scheme and it is isomorphic to  $\mathbb{G}_m^4$ . The stabilizers of the action  $\mathbb{G}_m^4 \curvearrowright \mathbb{G}_m^8$  are trivial. It follows that the quotient stack  $[\mathbb{G}_m^8/\mathbb{G}_m^4]$  is finely represented by  $\mathbb{G}_m^8/\mathbb{G}_m^4$ . This gives us a dense open subset  $\mathcal{U}$  of  $\overline{M}_Q$  isomorphic to  $\mathbb{G}_m^4$  with a universal family which can be realized as the quotient of  $\left((\mathbb{P}^1)^3 \times \mathbb{G}_m^8, V\left(\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k\right)\right)$  under the  $\mathbb{G}_m^4$ -action on the coefficients  $c_{ijk}$ .  $\square$

**Definition 3.19.** Let  $((\mathbb{P}^1)^3 \times \mathcal{U}, \mathfrak{S})$  be the universal family over  $\mathcal{U}$  constructed in Lemma 3.18. Denote by  $\mathcal{U}_{\text{sm}}$  the open subset of  $\mathcal{U}$  parametrizing stable toric pairs  $((\mathbb{P}^1)^3, B)$  such that  $(B, \Delta|_B)$  is as in Proposition 3.14. Let  $\mathfrak{S}_{\text{sm}}$  be the pullback of  $\mathfrak{S}/\mathcal{U}$  along the inclusion  $\mathcal{U}_{\text{sm}} \hookrightarrow \mathcal{U}$ . Then  $(\mathfrak{S}_{\text{sm}}, (\frac{1+\epsilon}{2})(\Delta \times \mathcal{U}_{\text{sm}})|_{\mathfrak{S}_{\text{sm}}})/\mathcal{U}_{\text{sm}}$  is a family for the moduli functor  $\overline{\mathcal{M}}_{6\epsilon^2}$ , and hence we have an induced morphism  $f: \mathcal{U}_{\text{sm}} \rightarrow \overline{M}_{6\epsilon^2}/S_3$ . We define  $\overline{M}_{D_{1,6}}$  to be the closure of the image of  $f$ .

**Observation 3.20.** Let  $p$  be a point in  $\mathcal{U}_{\text{sm}}$  and let  $((\mathbb{P}^1)^3, B)$  be the corresponding stable toric pair. Then the fiber of  $f: \mathcal{U}_{\text{sm}} \rightarrow \overline{M}_{D_{1,6}}$  over  $f(p)$  is in bijection with the  $\text{Sym}(Q)$ -orbit of  $((\mathbb{P}^1)^3, B)$  (this follows from Proposition 3.14). In particular,  $f$  is a quasi-finite map. This, together with the fact that  $f$  is dominant, implies that  $\dim(\overline{M}_{D_{1,6}}) = \dim(\mathcal{U}_{\text{sm}}) = \dim(\overline{M}_Q) = 4$ .  $\overline{M}_{D_{1,6}}$  is irreducible because  $\mathcal{U}_{\text{sm}}$  is irreducible. In conclusion,  $\overline{M}_{D_{1,6}}$  is a 4-dimensional irreducible projective coarse moduli space whose points in a dense open subset parametrize isomorphism classes of  $\text{Bl}_3 \mathbb{P}^2$  together with the three undistinguished weighted pairs of lines (and hence Enriques surfaces with our choice of divisor after an appropriate  $\mathbb{Z}_2^2$ -cover).

#### 4. STABLE REPLACEMENT IN A ONE-PARAMETER FAMILY

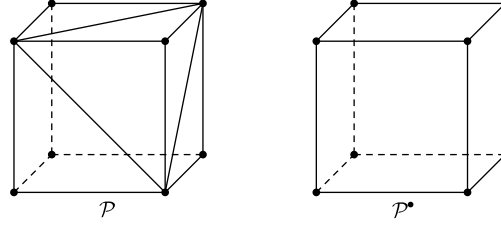
Let  $Q$  be the unit cube and let  $(A, \mathfrak{m})$  be a DVR. Let  $(\mathfrak{X}, \mathfrak{B})$  be a family of stable toric pairs of type  $\leq Q$  over  $\text{Spec}(A)$ . Define  $X = \mathfrak{X}_{\mathfrak{m}}$ ,  $B = \mathfrak{B}_{\mathfrak{m}}$ , and assume that the fiber of  $\mathfrak{X}$  over the generic point is isomorphic to  $(\mathbb{P}^1)_K^3$  where  $K$  is the field of fractions of  $A$ . Observe that the pair  $(B, (\frac{1+\epsilon}{2})\Delta|_B)$  may be not stable: to start with, the restriction  $\Delta|_B$  makes no sense if  $\Delta$  is not  $\mathbb{Q}$ -Cartier (see Proposition 5.2). In this section we define a new family  $(\mathfrak{X}^\bullet, \mathfrak{B}^\bullet)$  which is isomorphic to the original one in the complement of the central fiber, but the new central fiber  $(X^\bullet, B^\bullet)$  is such that  $(B^\bullet, (\frac{1+\epsilon}{2})\Delta^\bullet|_{B^\bullet})$  is a stable pair ( $\Delta^\bullet$  denotes the toric boundary of  $X^\bullet$ ). The stability of the pair  $(B^\bullet, (\frac{1+\epsilon}{2})\Delta^\bullet|_{B^\bullet})$  is proved later in Section 5.

**4.1. Families of stable toric pairs over a DVR.** Let  $(A, \mathfrak{m})$  be a DVR with field of fractions  $K$ , uniformizing parameter  $t$ , and residue field our fixed base field  $\mathbb{k}$ . Let  $T$  be a torus over  $\mathbb{k}$  with character lattice  $M$ . Let  $Q \subset M_{\mathbb{R}}$  be a lattice polytope. Define

$$\theta = \sum_{m \in Q \cap M} c_m(t) t^{h(m)} x^{(1,m)},$$

where, for all  $m \in Q \cap M$ ,  $c_m(t) \in A$ ,  $c_m(0) \in \mathbb{k}^*$ , and  $h(m) \in \mathbb{Z}$ . Observe that the association  $m \mapsto h(m)$  gives a height function  $h: Q \cap M \rightarrow \mathbb{Z}$ . Let  $Q^+ \subset M_{\mathbb{R}} \oplus \mathbb{R}$  be the convex hull of the half-lines  $(m, h(m) + \mathbb{R}_{\geq 0})$ ,  $m \in Q \cap M$ , and let  $\text{Cone}(Q^+) \subset \mathbb{R} \oplus M_{\mathbb{R}} \oplus \mathbb{R}$  be the cone over  $(1, Q^+)$  with vertex at the origin. Then  $h$  defines the following  $(\mathbb{Z} \oplus M)$ -graded  $A$ -algebra:

$$R = A[t^\ell x^{(d,m)} \mid (d, m, \ell) \in \text{Cone}(Q^+) \cap (\mathbb{Z} \oplus M \oplus \mathbb{Z})].$$

FIGURE 1. Modification  $\mathcal{P}^\bullet$  of the polyhedral subdivision  $\mathcal{P}$ 

Observe that  $\theta \in R$  is an element of  $\mathbb{Z}$ -degree 1. Let  $(\mathfrak{X}, \mathfrak{B})/\mathrm{Spec}(A)$  be the family of stable toric pairs associated to  $(R, \theta)$  (see Remark 2.7). If  $\eta$  is the generic point of  $\mathrm{Spec}(A)$ , then  $\mathfrak{X}_\eta = Y \times \mathrm{Spec}(K)$ , where  $Y$  is the toric variety associated to the polytope  $Q$ . The central fiber  $(\mathfrak{X}_\mathfrak{m}, \mathfrak{B}_\mathfrak{m})$  is a stable toric pair whose corresponding polyhedral subdivision of  $(Q, Q \cap M)$  is induced by the height function  $h$  (hence, it is a regular subdivision). The equation of  $\mathfrak{B}_\mathfrak{m}$  is given by

$$\sum_{m \in Q \cap M} c_m(0) x^{(1, m)} = 0.$$

For more details about this construction we refer to [A02, Section 2.8].

#### 4.2. Corner cuts.

**Definition 4.1.** Let  $Q$  be the unit cube. We call a *corner cut* the convex hull in  $Q$  of the points  $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$  up to symmetries of  $Q$ . An example of corner cut can be found in Figure 1<sup>2</sup> on the left. We call *apex* the vertex of a corner cut which is at the intersection of three edges of the cube.

**Notation 4.2.** Let  $\mathcal{P}$  be a polyhedral subdivision of a lattice polytope  $Q$ . We denote by  $\mathcal{P}_i$  the set of  $i$ -dimensional faces in  $\mathcal{P}$ .

**Definition 4.3.** Let  $\mathcal{P}$  be a polyhedral subdivision of  $(Q, Q \cap \mathbb{Z}^3)$ . We define a polyhedral subdivision  $\mathcal{P}^\bullet$  of  $(Q, Q \cap \mathbb{Z}^3)$  via the following algorithm:

- (1)  $\mathcal{R} = \mathcal{P}$ ;
- (2) If  $\mathcal{R}$  contains no corner cut, define  $\mathcal{P}^\bullet = \mathcal{R}$  and stop. Otherwise, go to step (3);
- (3) Let  $P \in \mathcal{R}$  be a corner cut and let  $R \in \mathcal{R}$  be that unique polytope sharing exactly a facet with  $P$ . Define

$$\mathcal{S} = (\mathcal{R}_3 \setminus \{P, R\}) \cup \{P \cup R\}.$$

Then redefine  $\mathcal{R}$  to be the polyhedral subdivision of  $Q$  generated by  $\mathcal{S}$ . Go to step (2).

In Figure 1 we give an explicit example of  $\mathcal{P}^\bullet$  given  $\mathcal{P}$ .

**4.3. The modified family  $(\mathfrak{X}^\bullet, \mathfrak{B}^\bullet)$ .** Let us adopt the same notation introduced in Section 4.1, but set  $Q$  to be the unit cube. We define another  $(\mathbb{Z} \oplus M)$ -graded  $A$ -algebra induced by  $\theta$  as follows. Denote by  $\mathcal{P}$  the regular polyhedral subdivision of  $(Q, Q \cap \mathbb{Z}^3)$  associated to  $(\mathfrak{X}_\mathfrak{m}, \mathfrak{B}_\mathfrak{m})$ . Assume  $P \in \mathcal{P}_3$  is a corner cut and let  $P' \in \mathcal{P}$  be that unique polytope sharing exactly a facet with  $P$ . Denote by  $L$  the unique hyperplane in  $\mathbb{R}^3 \oplus \mathbb{R}$  containing the points  $(m, h(m))$  for  $m \in P' \cap \mathbb{Z}^3$ . If  $m$  is the apex of the corner cut  $P$ , then there exists a unique positive rational number  $q_m$  such

<sup>2</sup>All the figures in this paper were realized using the software GeoGebra, Copyright ©International GeoGebra Institute, 2013.

that  $(m, h(m) - q_m) \in L$ . Moreover, up to a ramified finite base change, we can assume that  $q_m$  is integral. Let us consider the height function

$$h^\bullet(m) = \begin{cases} h(m) - q_m & \text{if } m \text{ is the apex of a corner cut,} \\ h(m) & \text{otherwise.} \end{cases}$$

Define a new  $(\mathbb{Z} \oplus M)$ -graded  $A$ -algebra  $R^\bullet$  as we did in Section 4.1, but using  $h^\bullet$  in place of  $h$ . Observe that  $R \subset R^\bullet$  is a degree preserving embedding of graded algebras. Therefore  $\theta \in R^\bullet$  is an element of  $\mathbb{Z}$ -degree 1 and the pair  $(R^\bullet, \theta)$  corresponds to a family  $\mathfrak{X}^\bullet / \text{Spec}(A)$  of stable toric varieties together with a Cartier divisor  $\mathfrak{B}^\bullet \subset \mathfrak{X}^\bullet$  given by the vanishing of  $\theta$ . Observe that  $(\mathfrak{X}_\eta, \mathfrak{B}_\eta)$  and  $((\mathfrak{X}^\bullet)_\eta, (\mathfrak{B}^\bullet)_\eta)$  are isomorphic over  $\text{Spec}(K)$  by construction. The central fiber  $((\mathfrak{X}^\bullet)_m, (\mathfrak{B}^\bullet)_m)$  is not a stable toric pair if and only if  $\mathcal{P}$  contains corner cuts. However,  $((\mathfrak{X}^\bullet)_m, \mathcal{O}((\mathfrak{B}^\bullet)_m))$  is a polarized stable toric variety whose corresponding polyhedral subdivision of  $(Q, Q \cap \mathbb{Z}^3)$  is  $\mathcal{P}^\bullet$  (see Definition 4.3). The equation of  $(\mathfrak{B}^\bullet)_m$  is given by

$$\sum_{\substack{m \in Q \cap M, \\ m \text{ is not an apex}}} c_m(0) x^{(1, m)} = 0.$$

Observe that if  $\mathcal{P}$  has no corner cuts, then  $(\mathfrak{X}, \mathfrak{B}) = (\mathfrak{X}^\bullet, \mathfrak{B}^\bullet)$ .

**Remark 4.4.** With the same notation introduced above, denote by  $(X, B)$  (resp.  $(X^\bullet, B^\bullet)$ ) the central fiber of  $(\mathfrak{X}, \mathfrak{B})$  (resp.  $(\mathfrak{X}^\bullet, \mathfrak{B}^\bullet)$ ). Then  $(X^\bullet, B^\bullet)$  only depends on  $(X, B)$  and it is independent from the whole family  $(\mathfrak{X}, \mathfrak{B}) / \text{Spec}(A)$ .

**Definition 4.5.** Let  $(X, B)$  be a stable toric pair of type  $\leq Q$ . Define  $(X^\bullet, B^\bullet)$  to be the central fiber of  $(\mathfrak{X}^\bullet, \mathfrak{B}^\bullet)$ , where  $(\mathfrak{X}, \mathfrak{B})$  is any one-parameter family with central fiber  $(X, B)$  and smooth generic fiber (such a family exists because  $\overline{M}_Q$  is irreducible, hence  $(X, B)$  is smoothable).  $(X^\bullet, B^\bullet)$  is well defined by Remark 4.4. Observe that, if  $\mathcal{P}$  has no corner cuts, then  $(X, B) = (X^\bullet, B^\bullet)$ .

## 5. ANALYSIS OF STABILITY

**5.1. Preliminaries.** Throughout this section let  $Q$  be the unit cube. We show that if  $(X, B)$  is a stable toric pair of type  $\leq Q$ , then  $(B^\bullet, (\frac{1+\epsilon}{2}) \Delta^\bullet|_{B_P^\bullet})$  is stable. Before formally stating this result, we need some preliminaries.

**Notation 5.1.** Consider a stable toric pair  $(X, B)$  of type  $\leq Q$  and let  $\mathcal{P}$  be the corresponding polyhedral subdivision of  $(Q, Q \cap \mathbb{Z}^3)$ . Let  $\coprod_{P \in \mathcal{P}_3} X_P \rightarrow X$  be the normalization of  $X$ , where  $X_P$  is the toric variety corresponding to the polytope  $P$ . Then we denote by  $\Delta_P$  the toric boundary of  $X_P$ , by  $D_P \subset X_P$  the conductor divisor, and by  $B_P$  the restriction to  $X_P$  of the preimage of  $B$  under the normalization morphism. Define  $X_P^\bullet, \Delta_P^\bullet, D_P^\bullet, B_P^\bullet$  analogously with  $(X^\bullet, B^\bullet)$  in place of  $(X, B)$ .

**Proposition 5.2.** *Let  $(X, B)$  be a stable toric pair of type  $\leq Q$  and let  $\mathcal{P}$  be the associated polyhedral subdivision of  $(Q, Q \cap \mathbb{Z}^3)$ . If  $\mathcal{P}$  does not contain a corner cut, then  $\Delta$  is Cartier. If  $\mathcal{P}$  contains a corner cut, then  $\Delta$  is not  $\mathbb{Q}$ -Cartier.*

*Proof.* Define a piecewise linear function on the normal fan of each maximal dimensional polytope  $P \in \mathcal{P}$  as follows: associate to the  $\mathbb{Z}$ -generators of the rays, negative the lattice distance between the facet of  $2P$  normal to the ray and the lattice point  $(1, 1, 1)$ . Observe that this number is 0 if the facet contains  $(1, 1, 1)$ , and  $-1$  otherwise. If  $\mathcal{P}$  has no corner cuts, then this gives a Cartier divisor on  $X$  equal to  $\Delta$ .

Now assume  $\mathcal{P}$  contains a corner cut  $P$ . Denote with  $R$  that unique polytope in  $\mathcal{P}$  sharing exactly a facet with  $P$ . Let  $\ell_1, \ell_2, \ell_3$  be the three edges of  $P$  which do not contain the apex. Observe that  $\ell_i$  can be contained in two or three maximal dimensional polytopes in  $\mathcal{P}$  ( $P$  and  $R$  included),  $i = 1, 2, 3$ .

If some  $\ell_i$  is contained in three maximal dimensional polytopes, then take a point  $x \in X$  lying on the torus invariant line corresponding to  $\ell_i$ . If  $\Delta$  is  $\mathbb{Q}$ -Cartier, then  $m\Delta$  is given by the vanishing of one equation in an open neighborhood of  $x$  for some  $m > 0$ . However, the vanishing locus of this equation on  $X_R$  has codimension 2, which cannot be.

Assume that each  $\ell_i$  is only contained in  $P$  and  $R$ . Denote by  $\nu: X^\nu \rightarrow X$  the normalization. If  $\Delta$  is  $\mathbb{Q}$ -Cartier, then  $(\nu^*\Delta)|_{X_R} = \Delta_R - D_R$  is also  $\mathbb{Q}$ -Cartier. But this is a contradiction because there is no  $\mathbb{Q}$ -piecewise linear function on the normal fan of  $R$  corresponding to  $\Delta_R - D_R$  (consider the normal cone to a vertex of  $R$  in common with  $P$ ).  $\square$

**Theorem 5.3.** *Let  $Q$  be the unit cube and let  $(X, B)$  be a stable toric pair of type  $\leq Q$ . Consider  $(X^\bullet, B^\bullet)$  as in Definition 4.5. Then  $(B^\bullet, (\frac{1+\epsilon}{2})\Delta^\bullet|_{B^\bullet})$  is a stable pair (it makes sense to consider  $\Delta^\bullet|_{B^\bullet}$  by Proposition 5.2).*

**5.2. Proof of Theorem 5.3.** Let  $\mathcal{P}$  be the polyhedral subdivision of  $(Q, Q \cap \mathbb{Z}^3)$  associated to  $(X, B)$ . We show that for all  $P \in \mathcal{P}^\bullet$ , the pair  $(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + (\frac{1+\epsilon}{2})(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet})$  is stable. It is easy to observe that there are four possibilities for  $P$  up to symmetries of  $Q$  as shown in Figure 2.

**Definition 5.4.** We say that  $P \in \mathcal{P}^\bullet$  has type (a) (resp. (b), (c), (d)) if  $P$  is equal to the polytope in Figure 2(a) (resp. (b), (c), (d)) up to symmetries of  $Q$ .

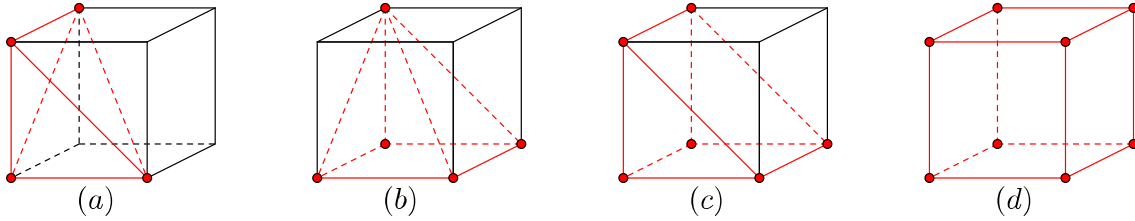


FIGURE 2. Possible maximal dimensional polytopes in  $\mathcal{P}^\bullet$  up to symmetries of  $Q$

**Proposition 5.5.** *Given  $P$  of type (a), then the pair  $(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + (\frac{1+\epsilon}{2})(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet})$  is stable.*

*Proof.*

$$\begin{aligned} X_P^\bullet &= \mathbb{P}^3 \ni [W_0 : \dots : W_3], \\ D_P^\bullet &= V(W_0) + V(W_1), \\ \Delta_P^\bullet - D_P^\bullet &= V(W_2) + V(W_3), \\ B_P^\bullet &= V(a_0W_0 + \dots + a_3W_3), \end{aligned}$$

where  $a_i \neq 0$  for all  $i = 0, \dots, 3$ . To find the equation of  $B_P^\bullet$  we used Remark 2.5, and  $a_i \neq 0$  for all  $i$  because there is no corner cut that can possibly be contained in  $P$  (see the construction of  $(X^\bullet, B^\bullet)$  in Section 4.3). Then  $B_P^\bullet$  is isomorphic to  $\mathbb{P}^2$  and  $\Delta_P^\bullet$  restricts to  $B_P^\bullet$  giving four lines in general

linear position, implying that  $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$  is log canonical. Finally, if  $L$  denotes a generic line in  $B_P^\bullet$ , then

$$K_{B_P^\bullet} + D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet} \sim \epsilon L,$$

which is ample.  $\square$

The next lemma is used in the analysis of the cases  $P$  of type (b), (c), (d).

**Lemma 5.6.** *Let  $L_1, L_2, L_3$  be three distinct lines in  $\mathbb{A}^2$  through the point  $(0,0)$ . Then the pair  $(\mathbb{A}^2, \left(\frac{1+\epsilon}{2}\right)(L_1 + L_2 + L_3))$  is log canonical.*

*Proof.* Consider the blow up morphism  $\sigma: \text{Bl}_{(0,0)} \mathbb{A}^2 \rightarrow \mathbb{A}^2$  and let  $E$  be the exceptional divisor. Then we have that

$$\begin{aligned} K_{\text{Bl}_{(0,0)} \mathbb{A}^2} + \sigma_*^{-1} \left( \left( \frac{1+\epsilon}{2} \right) \sum_{i=1}^3 L_i \right) &= \sigma^* \left( K_{\mathbb{A}^2} + \left( \frac{1+\epsilon}{2} \right) \sum_{i=1}^3 L_i \right) + aE \Rightarrow \\ E &= 3 \left( \frac{1+\epsilon}{2} \right) E + aE \Rightarrow \\ a &= -\frac{1+3\epsilon}{2} > -1. \end{aligned}$$

Therefore,  $(\mathbb{A}^2, \left(\frac{1+\epsilon}{2}\right)(L_1 + L_2 + L_3))$  is log canonical because  $\sigma$  is a log resolution.  $\square$

**Proposition 5.7.** *Given  $P$  of type (b), then the pair  $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$  is stable.*

*Proof.*

$$\begin{aligned} X_P^\bullet &= V(W_0W_1 - W_2W_3) \subset \mathbb{P}^4 \ni [W_0 : \dots : W_4], \\ D_P^\bullet &= V(W_0, W_2) + V(W_0, W_3) = V(W_0W_1 - W_2W_3, W_0), \\ \Delta_P^\bullet - D_P^\bullet &= V(W_1, W_2) + V(W_1, W_3) + V(W_0W_1 - W_2W_3, W_4) \\ &= V(W_0W_1 - W_2W_3, W_1) + V(W_0W_1 - W_2W_3, W_4), \\ B_P^\bullet &= V(a_0W_0 + \dots + a_4W_4) \cap X_P^\bullet, \end{aligned}$$

where  $a_i \neq 0$  for  $i = 1, \dots, 4$  and  $a_0$  can possibly vanish (we can have at most one corner cut contained in  $P$ ).  $B_P^\bullet \cong \mathbb{P}^1 \times \mathbb{P}^1$  because it is a hyperplane section of the projective cone  $X_P^\bullet$  which does not pass through the vertex.

Now let us study the restrictions of  $D_P^\bullet$  and  $\Delta_P^\bullet - D_P^\bullet$  to  $B_P^\bullet$ . This boils down to understand how the coordinate hyperplanes  $H_i = V(W_i)$ ,  $i = 0, \dots, 4$ , restrict to  $B_P^\bullet$ . First of all, observe that  $H_i$  cuts on  $B_P^\bullet$  a curve  $C$  of divisor class  $(1,1)$ . To show this, denote by  $(a,b)$  the divisor class of  $C = B_P^\bullet \cap H_i$ . If  $H' = V(a_0W_0 + \dots + a_4W_4)$  and  $H$  is a generic hyperplane in  $\mathbb{P}^4$ , then the self-intersection of  $C$  is given by

$$C^2 = H_i|_{B_P^\bullet} \cdot H_i|_{B_P^\bullet} = H_i \cdot H_i \cdot B_P^\bullet = H_i \cdot H_i \cdot X_P^\bullet \cdot H' = H_i \cdot H_i \cdot 2H \cdot H' = 2.$$

On the other hand,  $C^2 = (a,b)^2 = 2ab = 2$ , implying that  $(a,b) = (1,1)$ . For  $i \neq 4$ ,  $H_i \cap B_P^\bullet$  is always reducible, so it is given by two curves of divisor classes  $(1,0)$  and  $(0,1)$ . If  $a_0 \neq 0$ , then it is easy to check that  $H_4 \cap B_P^\bullet$  is smooth for a generic choice of the coefficients, but it can possibly



break into two curves. In any case,  $\Delta_P^\bullet$  restricts to  $B_P^\bullet$  giving a simple normal crossing divisor, implying that  $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$  is log canonical.

If  $a_0 = 0$ , then  $H_4 \cap B_P^\bullet$  is irreducible, but it passes through the singular point of  $H_1 \cap B_P^\bullet$ . However, we have that  $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$  is log canonical by Lemma 5.6.

Finally, observe that

$$K_{B_P^\bullet} + D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet} \sim (-2, -2) + (1, 1) + \left(\frac{1+\epsilon}{2}\right)(2, 2) = \epsilon(1, 1),$$

which is ample.  $\square$

In the cases  $P$  of type (c) or (d), the ideas are similar to the ones we already used in the previous propositions. These remaining cases are in the Appendix (see Propositions 10.6 and 10.8).  $\square$

**Remark 5.8.** From the proof of Theorem 5.3 we have an explicit description of all the possible  $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$  for any stable toric pair  $(X, B)$  of type  $\leq Q$ . Here we want to summarize these possibilities. In Figure 3, a triangle (resp. trapezoid, parallelogram) means  $B_P^\bullet \cong \mathbb{P}^2$  (resp.  $\mathbb{F}^1$ ,  $\mathbb{P}^1 \times \mathbb{P}^1$ ).  $D_P^\bullet|_{B_P^\bullet}$  is represented by the thickened segments and  $(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}$  by the colored segments. First, let us assume that  $\mathcal{P}$  has no corner cuts, so that  $(X, B) = (X^\bullet, B^\bullet)$ . In all the cases that follow,  $\Delta_P|_{B_P}$  is simple normal crossing.

- (a)  $B_P \cong \mathbb{P}^2$  and  $D_P|_{B_P}$  (resp.  $(\Delta_P - D_P)|_{B_P}$ ) consists of two lines;
- (b)  $B_P \cong \mathbb{P}^1 \times \mathbb{P}^1$  and  $D_P|_{B_P}$  consists of two incident rulings.  $(\Delta_P - D_P)|_{B_P}$  is given by two incident rulings and a curve of divisor class  $(1, 1)$  which can possibly be reducible;
- (c1)  $B_P \cong \mathbb{F}_1$  and  $D_P|_{B_P}$  is a line disjoint from the exceptional divisor.  $(\Delta_P - D_P)|_{B_P}$  is given by two fibers and two lines disjoint from the exceptional divisor. Exactly one of these last two lines can possibly break into the union of the exceptional divisor and a fiber;
- (c2)  $B_P \cong \mathbb{F}_1$  and  $D_P|_{B_P}$  is the union of the exceptional divisor and a fiber.  $(\Delta_P - D_P)|_{B_P}$  is given by two fibers and two lines disjoint from the exceptional divisor;
- (c3)  $B_P$  is isomorphic to the union of  $\mathbb{P}^2$  and  $\mathbb{P}^1 \times \mathbb{P}^1$  glued along a line in  $\mathbb{P}^2$  and a ruling in  $\mathbb{P}^1 \times \mathbb{P}^1$ .  $D_P|_{B_P}$  consists of a line in  $\mathbb{P}^2$  and a ruling in  $\mathbb{P}^1 \times \mathbb{P}^1$ .  $(\Delta_P - D_P)|_{B_P}$  is given by two lines on the  $\mathbb{P}^2$  component and four rulings on  $\mathbb{P}^1 \times \mathbb{P}^1$  arranged as shown in Figure 3(c3);
- (d1)  $B_P \cong \text{Bl}_3 \mathbb{P}^2$  and  $D_P|_{B_P} = \emptyset$ .  $\Delta_P|_{B_P}$  is as in Proposition 3.8 (this is the generic case), or some lines can possibly break into two intersecting  $(-1)$ -curves;
- (d1')  $B_P$  is a singular del Pezzo surface of degree 6 with exactly one  $A_1$  singularity. This singularity can lie on at most one irreducible component of  $\Delta_P|_{B_P}$ ;
- (d2)  $B_P$  is isomorphic to the union of two copies of  $\mathbb{P}^1 \times \mathbb{P}^1$  glued along a ruling and an irreducible curve of divisor class  $(1, 1)$ .  $\Delta_P|_{B_P}$  is given by four rulings on one component and six rulings on the other. These are arranged as shown in Figure 3(d2);
- (d3)  $B_P$  is isomorphic to the union of three copies of  $\mathbb{P}^1 \times \mathbb{P}^1$  glued along rulings as shown in Figure 3(d3).  $\Delta_P|_{B_P}$  consists of four rulings on each component as shown in Figure 3(d3).

Now assume that  $\mathcal{P}$  has a corner cut and let  $P \in \mathcal{P}^\bullet$ . In this case, the possibilities for the pair  $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$  are as above, with the difference that now  $(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}$  is allowed to have triple intersection points.

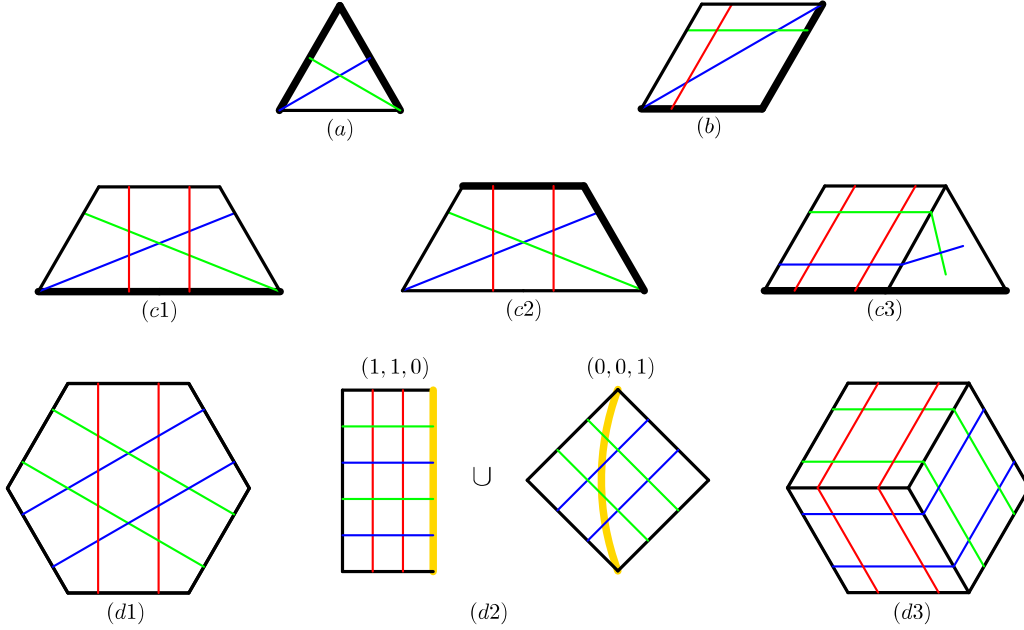


FIGURE 3. The pictures represent  $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$  for  $P \in \mathcal{P}^\bullet$  (see Remark 5.8). The restriction  $(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}$  is in blue, green and red,  $D_P^\bullet|_{B_P^\bullet}$  is thickened. Two irreducible components are glued together along the yellow curves

## 6. KSBA COMPACTIFICATION FOR $D_{1,6}$ -POLARIZED ENRIQUES SURFACES

**6.1. Map to the KSBA compactification.** In Theorem 6.4 we describe a surjective (algebraic) morphism  $\overline{M}_Q^\nu \rightarrow \overline{M}_{D_{1,6}}$  which gives a geometric interpretation of all the  $\mathbb{k}$ -points of  $\overline{M}_{D_{1,6}}$  (the global geography of  $\overline{M}_{D_{1,6}}$  is described in Section 6.2). We need some preliminaries from [GG].

**Notation 6.1.** Consider a map  $g: \text{Spec}(K) \rightarrow Y$ , where  $K$  is the field of fractions of a DVR  $(A, \mathfrak{m})$  and  $Y$  is a proper scheme. By the valuative criterion of properness,  $g$  extends in a unique way to a map  $\overline{g}: \text{Spec}(A) \rightarrow Y$ . We denote by  $\lim g$  the point  $\overline{g}(\mathfrak{m})$ .

**Theorem 6.2** ([GG, Theorem 7.3]). *Suppose  $X_1$  and  $X_2$  are proper schemes over a noetherian scheme  $S$  with  $X_1$  normal. Let  $U \subseteq X_1$  be an open dense set and  $f: U \rightarrow X_2$  an  $S$ -morphism. Then  $f$  extends to an  $S$ -morphism  $\overline{f}: X_1 \rightarrow X_2$  if, and only if, for any DVR  $(A, \mathfrak{m})$  and any morphism  $\text{Spec}(K) \rightarrow U$ , the point  $\lim(f \circ g)$  of  $X_2$  is uniquely determined by the point  $\lim g$  of  $X_1$ .*

**Notation 6.3.** For a scheme  $X$ , let  $X^\nu$  be the normalization of  $X$  with the reduced scheme structure.

**Theorem 6.4.** *Let  $Q$  be the unit cube. Then there is a surjective (algebraic) morphism  $\overline{M}_Q^\nu \rightarrow \overline{M}_{D_{1,6}}$  which is given on  $\mathbb{k}$ -points by*

$$(X, B) \mapsto \left(B^\bullet, \left(\frac{1+\epsilon}{2}\right) \Delta^\bullet|_{B^\bullet}\right).$$

*Proof.* Set  $X_1 = \overline{M}_Q^\nu$ ,  $X_2 = \overline{M}_{D_{1,6}}$ ,  $U = \mathcal{U}_{\text{sm}}$  (which was constructed in Definition 3.19), and let  $S = \text{Spec}(\mathbb{k})$ . The claimed morphism  $X_1 \rightarrow X_2$  is obtained by extending  $f: \mathcal{U}_{\text{sm}} \rightarrow X_2$  to the whole  $X_1$  by means of Theorem 6.2.

Let  $(A, \mathfrak{m})$  be any DVR and consider a map  $g: \text{Spec}(K) \rightarrow \mathcal{U}_{\text{sm}}$ . The point  $\lim g \in X_1$  corresponds to a stable toric pair  $(X, B)$  of type  $\leq Q$ . Denote by  $\Delta$  the toric boundary of  $X$ . If we prove that  $\lim(f \circ g)$  corresponds to the stable pair  $(B^\bullet, (\frac{1+\epsilon}{2}) \Delta^\bullet|_{B^\bullet})$ , then we are done by Theorem 6.2, because this shows that  $\lim(f \circ g)$  only depends on  $\lim g$ , and not from the way we approach it (see Remark 4.4).

Let  $((\mathbb{P}^1)^3 \times \mathcal{U}_{\text{sm}}, \mathfrak{S}_{\text{sm}})/\mathcal{U}_{\text{sm}}$  be as in Definition 3.19. Let  $((\mathbb{P}_K^1)^3, \mathfrak{B}^\circ)$  be the pullback of  $((\mathbb{P}^1)^3 \times \mathcal{U}_{\text{sm}}, \mathfrak{S}_{\text{sm}})$  under the map  $g$  and denote by  $(\mathfrak{X}, \mathfrak{B})$  its completion over  $\text{Spec}(A)$  (or a finite ramified base change of it). Consider  $(\mathfrak{X}^\bullet, \mathfrak{B}^\bullet)$  and, if  $D$  denotes the toric boundary of  $(\mathbb{P}^1)^3$ , let  $\overline{D}_K$  be the closure of  $D_K = D \times \text{Spec}(K)$  in  $\mathfrak{X}^\bullet$ . Then the central fiber of  $(\mathfrak{B}^\bullet, (\frac{1+\epsilon}{2}) \overline{D}_K|_{\mathfrak{B}^\bullet})$ , which is  $(B^\bullet, (\frac{1+\epsilon}{2}) \Delta^\bullet|_{B^\bullet})$  by construction, is the stable pair corresponding to  $\lim(f \circ g)$ .  $\square$

**Theorem 6.5.** *Let  $\overline{M}_Q^\nu \rightarrow \overline{M}_{D_{1,6}}$  be the morphism in Theorem 6.4. Then the induced morphism  $\overline{M}_Q^\nu/\text{Sym}(Q) \rightarrow \overline{M}_{D_{1,6}}^\nu$  is an isomorphism.*

*Proof.* The group  $\text{Sym}(Q)$  acts on  $\overline{M}_Q^\nu$  because  $\overline{M}_Q^\nu$  is the projective toric variety associated to the secondary polytope of  $(Q, Q \cap \mathbb{Z}^3)$ . The modular interpretation of the action is the following:  $\text{Sym}(Q)$  acts on the stable toric pair  $(X, B)$  by changing the torus action on it. In particular, the morphism  $\overline{M}_Q^\nu \rightarrow \overline{M}_{D_{1,6}}$  is  $\text{Sym}(Q)$ -equivariant. Therefore, we have an induced morphism  $\overline{M}_Q^\nu/\text{Sym}(Q) \rightarrow \overline{M}_{D_{1,6}}$  which lifts to the normalization of  $\overline{M}_{D_{1,6}}$ .

The fibers of the restriction of  $\overline{M}_Q^\nu \rightarrow \overline{M}_{D_{1,6}}$  to  $\mathcal{U}_{\text{sm}}$  are exactly  $\text{Sym}(Q)$ -orbits (see Definition 3.19 and Observation 3.20), implying that the morphism  $\overline{M}_Q^\nu/\text{Sym}(Q) \rightarrow \overline{M}_{D_{1,6}}^\nu$  is generically bijective. Then the fact that  $\overline{M}_Q^\nu/\text{Sym}(Q) \rightarrow \overline{M}_{D_{1,6}}^\nu$  is an isomorphism follows from the Zariski Main Theorem (see [M99, Chapter III, Section 9]) after we show that  $\overline{M}_Q^\nu \rightarrow \overline{M}_{D_{1,6}}$  is quasi-finite.

It is enough to prove that no 1-dimensional boundary stratum of  $\overline{M}_Q^\nu$  is contracted by  $\overline{M}_Q^\nu \rightarrow \overline{M}_{D_{1,6}}$ . These strata correspond to the minimal elements of the poset of regular polyhedral subdivisions of  $(Q, Q \cap \mathbb{Z}^3)$  which are not triangulations (all the polyhedral subdivisions of  $(Q, Q \cap \mathbb{Z})$  are regular, see the proof of Proposition 3.16). If  $\mathcal{P}$  is one of these polyhedral subdivisions, then it has to contain a subpolytope of  $Q$  with vertices in  $Q \cap \mathbb{Z}^3$  which can be subdivided further only once. An easy enumeration shows that  $\mathcal{P}$  has to contain one of the polytopes listed in Figure 4. Denote by  $P$  one of such polytopes, and let  $(X, B)$  be any stable toric pair with  $\mathcal{P}$  as corresponding polyhedral subdivision of  $(Q, Q \cap \mathbb{Z}^3)$ . Then, as  $(X, B)$  varies among the stable toric pairs parametrized by the 1-dimensional boundary stratum corresponding to  $\mathcal{P}$ ,  $(B^\bullet, (\frac{1+\epsilon}{2}) \Delta^\bullet|_{B^\bullet})$  describes a 1-dimensional family of stable pairs. To prove this, if  $P^\bullet \in \mathcal{P}^\bullet$  is the polytope corresponding to  $P \in \mathcal{P}$ , it is enough to consider the irreducible component of  $(B^\bullet, (\frac{1+\epsilon}{2}) \Delta^\bullet|_{B^\bullet})$  corresponding to  $P^\bullet$ .  $\square$

## 6.2. Description of the boundary of $\overline{M}_{D_{1,6}}^\nu$ .

**Definition 6.6.** The *boundary* of the moduli space  $\overline{M}_{D_{1,6}}^\nu$  is the closed subset whose  $\mathbb{k}$ -points parametrize stable pairs  $(B, D)$  with  $B$  reducible. Let  $(B, D)$  be a stable pair parametrized by the boundary, and consider the locus of points in  $\overline{M}_{D_{1,6}}^\nu$  parametrizing stable pairs  $(B', D')$  such that  $B \cong B'$ . We call the closure of such locus a *stratum*.

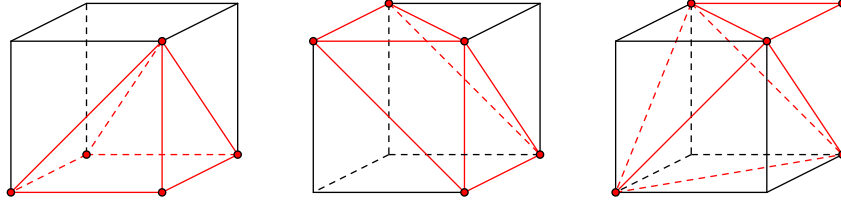


FIGURE 4. Subpolytopes of  $Q$  which can be subdivided further only once (see the proof of Theorem 6.5)

**Theorem 6.7.** *The boundary of  $\overline{M}_{D_{1,6}}^\nu$  is stratified as shown in Figure 6. The strata are organized from bottom to top in increasing order of dimension. Going from left to right in Figure 6, call the 0-dimensional strata even, odd of type 1, and odd of type 2 (this terminology is borrowed from [O]). Then the strata containing the even 0-dimensional stratum correspond bijectively to the polyhedral subdivisions of  $(Q, Q \cap \mathbb{Z}^3)$  without corner cuts up to symmetries of  $Q$ , where  $Q$  is the unit cube.*

*Proof.* By Theorem 6.5 any stable pair parametrized by  $\overline{M}_{D_{1,6}}^\nu$  is in the form  $(B^\bullet, (\frac{1+\epsilon}{2}) \Delta^\bullet|_{B^\bullet})$  for some stable toric pair  $(X, B)$  parametrized by  $\overline{M}_Q^\nu$ . In Remark 5.8 we listed all the possibilities for the pairs  $(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + (\frac{1+\epsilon}{2}) (\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet})$ , which can be glued together for  $P \in \mathcal{P}^\bullet$  to recover  $(B^\bullet, (\frac{1+\epsilon}{2}) \Delta^\bullet|_{B^\bullet})$ , where  $\mathcal{P}^\bullet$  is a polyhedral subdivisions of  $(Q, Q \cap \mathbb{Z}^3)$  without corner cuts. These subdivisions are shown in Figure 5 up to symmetries of  $Q$ . The end result is shown in Figure 6, where for each stratum we choose the generic element to represent it. The three colors (blue, green and red) used to draw the divisor have the following meaning: lines sharing the same color come from the same pair of lines on the original  $\text{Bl}_3 \mathbb{P}^2$ . We want to remark that, even though we use three different colors, we do not distinguish the three pairs because we took the quotient by  $S_3$  in our definition of  $\overline{M}_{D_{1,6}}^\nu$ . The yellow color is used to indicate lines along which two irreducible components are glued together. The claimed combinatorial interpretation of the strata containing the even 0-dimensional stratum follows immediately after comparing Figure 5 and Figure 6.  $\square$

**6.3. Degenerations of Enriques surfaces.** After describing all the degenerations of  $\text{Bl}_3 \mathbb{P}^2$  together with the three undistinguished pairs of weighted lines parametrized by  $\overline{M}_{D_{1,6}}^\nu$ , we want to discuss what are the overlying  $\mathbb{Z}_2^2$ -covers, which correspond to actual degenerations of Enriques surfaces. We describe such  $\mathbb{Z}_2^2$ -covers for the stable pairs  $(B, D)$  parametrized by a general point in each boundary stratum of  $\overline{M}_{D_{1,6}}^\nu$ .

- (a) Consider the stable pair parametrized by the odd 0-dimensional stratum of type 2. Then its  $\mathbb{Z}_2^2$ -cover is isomorphic to the quotient of

$$V((X_0^2 - X_1^2)(Y_0^2 - Y_1^2)(Z_0^2 - Z_1^2)) \subset (\mathbb{P}^1)^3,$$

by the involution  $\iota$  (see Remark 3.15 and the proof of Proposition 10.8). This quotient consists of three copies of  $\mathbb{P}^1 \times \mathbb{P}^1$  glued along rulings in such a way that its dual complex gives a triangulation of the real projective plane.

- (b) The stable pair parametrized by a general point in the maximal 1-dimensional stratum of  $\overline{M}_{D_{1,6}}^\nu$  has  $\mathbb{Z}_2^2$ -cover isomorphic to the quotient of

$$V((X_0^2 - X_1^2)(Y_0^2 Z_0^2 + Y_1^2 Z_0^2 + Y_0^2 Z_1^2 + \lambda Y_1^2 Z_1^2)) \subset (\mathbb{P}^1)^3, \lambda \neq 0, 1,$$

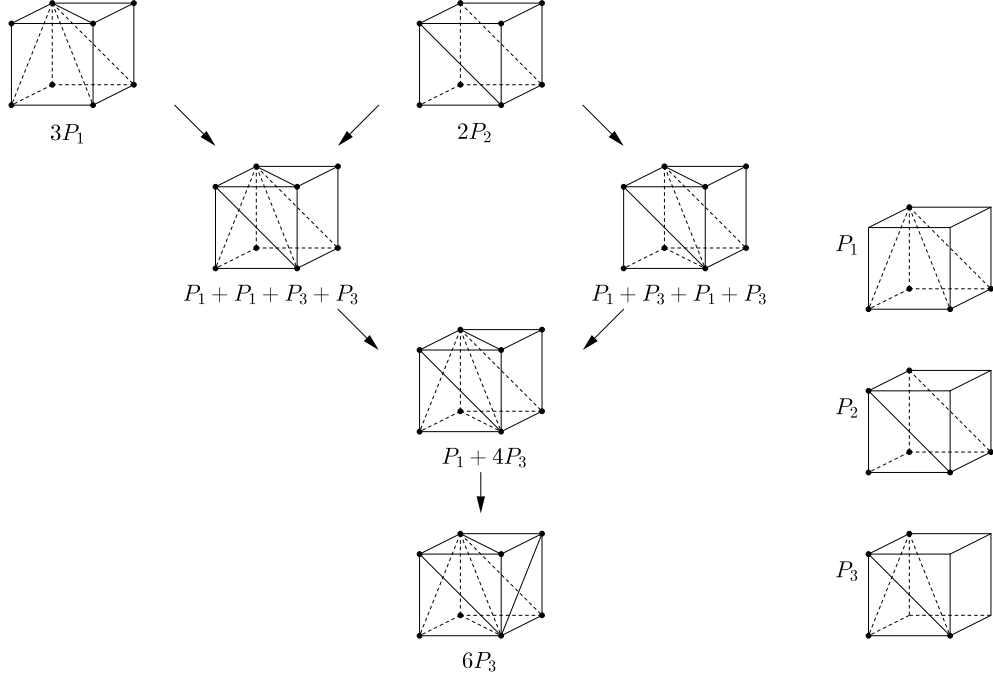


FIGURE 5. Polyhedral subdivisions of  $(Q, Q \cap \mathbb{Z}^3)$  without corner cuts up to symmetry and ordered by refinement

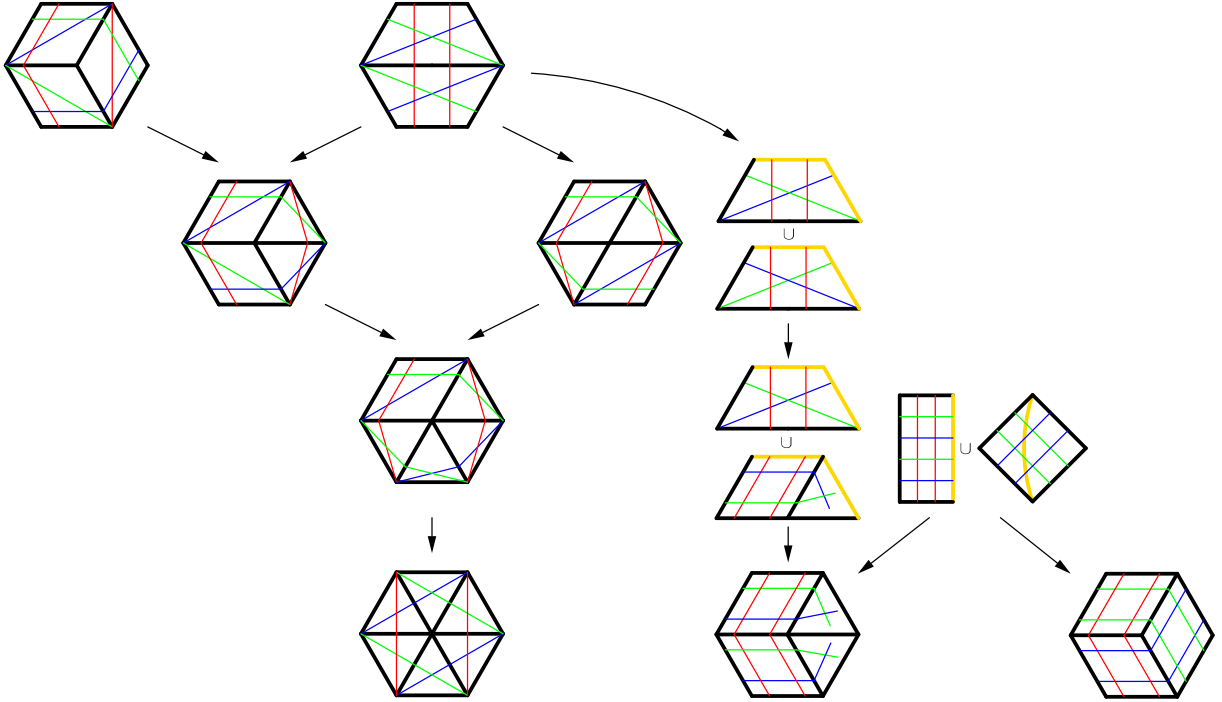


FIGURE 6. Stratification of the boundary of  $\overline{M}_{D_{1,6}}^\nu$

by  $\iota$  (again, see Remark 3.15 and the proof of Proposition 10.8). If we define

$$E = V(Y_0^2 Z_0^2 + Y_1^2 Z_0^2 + Y_0^2 Z_1^2 + \lambda Y_1^2 Z_1^2) \subset \mathbb{P}^1 \times \mathbb{P}^1,$$

then the irreducible components of this  $\mathbb{Z}_2^2$ -cover are a copy of  $\mathbb{P}^1 \times \mathbb{P}^1$ , and an elliptic fibration  $F$  over  $\mathbb{P}^1$  with fibers isomorphic to  $E$  and two double fibers. These surfaces are glued along  $E \subset \mathbb{P}^1 \times \mathbb{P}^1$  and a reduced fiber of  $F$ .

- (c) Consider the stable pair parametrized by a general point in the irreducible boundary divisor containing two 0-dimensional strata. Let us describe the  $\mathbb{Z}_2^2$ -cover  $X$  of one of the two irreducible components, which are both isomorphic to  $\mathbb{F}_1$ . Let  $h$  be a section of self-intersection 1 and  $f$  a fiber. Then the building data for the cover  $\pi: X \rightarrow \mathbb{F}_1$  is given by  $D_a \sim 2f + h, D_b \sim D_c \sim h$ , implying that

$$K_X \sim_{\mathbb{Q}} \pi^* \left( K_{\mathbb{F}_1} + \frac{1}{2}(D_a + D_b + D_c) \right) \sim_{\mathbb{Q}} -\frac{1}{2}\pi^*(h).$$

This shows that  $-K_X$  is big, nef, and  $K_X^2 = 1$ . Therefore,  $X$  is a weak del Pezzo surface of degree 1. The total degeneration is given by two of such weak del Pezzo surfaces glued along a genus 1 curve.

- (d) Let  $X$  be the  $\mathbb{Z}_2^2$ -cover of the  $\mathbb{P}^1 \times \mathbb{P}^1$  in Figure 3(b). Denote by  $\ell_1$  and  $\ell_2$  two incident rulings. Then the building data for the cover  $\pi: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is given by  $D_a \sim D_b \sim D_c \sim \ell_1 + \ell_2$ . From this we obtain that  $K_X \sim_{\mathbb{Q}} -\frac{1}{2}\pi^*(\ell_1 + \ell_2)$ . It follows that  $-K_X$  is ample and  $K_X^2 = 2$ . Hence,  $X$  is a del Pezzo surface of degree 2.
- (e) Let us describe the  $\mathbb{Z}_2^2$ -cover of the  $\mathbb{P}^2$  in Figure 3(a). If  $\ell$  denotes a line in  $\mathbb{P}^2$ , then the building data for the cover  $\pi: X \rightarrow \mathbb{P}^2$  is  $D_a \sim D_b \sim 2\ell, D_c \sim 0$ . Therefore,  $K_X \sim_{\mathbb{Q}} -\pi^*(\ell)$ , implying that  $-K_X$  is ample and  $K_X^2 = 4$ . Hence,  $X$  is a del Pezzo surface of degree 4.
- (f) Let us describe the  $\mathbb{Z}_2^2$ -cover of the  $\mathbb{P}^1 \times \mathbb{P}^1$ -component in Figure 3(c3). If  $\ell_1$  and  $\ell_2$  denote two incident rulings, then the building data for the cover  $\pi: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is given by  $D_a \sim 2\ell_1 + \ell_2, D_b \sim D_c \sim \ell_2$ . Therefore,  $K_X \sim_{\mathbb{Q}} -\frac{1}{2}\pi^*(2\ell_1 + \ell_2)$ ,  $-K_X$  is ample,  $K_X^2 = 4$ , and  $X$  is a del Pezzo surface of degree 4.

**Remark 6.8.** A Coble surface is a smooth rational projective surface  $X$  with  $|-K_X| = \emptyset$  and  $|-2K_X| \neq \emptyset$  (see [DZ]). These are related to our degenerations of Enriques surfaces as follows. Let  $S$  be the  $\mathbb{Z}_2^2$ -cover of  $\text{Bl}_3 \mathbb{P}^2$  branched along  $\sum_{i=1}^3 (\ell_i + \ell'_i)$  (see Notation 3.7), and assume that this lines configuration has exactly one triple intersection point. Then  $S$  has a quotient singularity of type  $\frac{(1,1)}{4}$  over this triple intersection point, and the minimal resolution  $\tilde{S}$  of  $S$  is a Coble surface. This follows from Castelnuovo rationality criterion, and from  $K_{\tilde{S}} = -\frac{1}{2}E$ , where  $E$  is the exceptional divisor over the quotient singularity. The Zariski closure in  $\overline{M}_{D_{1,6}}^{\nu}$  of the locus of points parametrizing these surfaces  $S$  defines a divisor.

## 7. RELATION WITH THE BAILY-BOREL COMPACTIFICATION

In what follows we construct a morphism from the KSBA compactification  $\overline{M}_{D_{1,6}}^{\nu}$  to the Baily-Borel compactification of  $\mathcal{D}/\Gamma$ , where  $\mathcal{D}$  is the period domain parametrizing  $D_{1,6}$ -polarized Enriques surfaces (details in Section 7.2). Section 7.1 contains a technical result which is fundamental to construct such morphism. In Section 7.3 we show that the combinatorics of the boundary strata of  $\overline{M}_{D_{1,6}}^{\nu}$  containing the even 0-dimensional stratum has another interpretation in terms of Vinberg diagrams. Let  $k = \mathbb{C}$  be our base field.

### 7.1. Generalized type of degenerations of stable K3 surface pairs.

**Remark 7.1.** In Section 7 our focus moves from Enriques surfaces to K3 surfaces. The reason for this is that in Theorem 7.8 we compute limits of one-parameter families of  $D_{1,6}$ -polarized Enriques surfaces in the Baily-Borel compactification of  $\mathcal{D}/\Gamma$ . This is done by using the corresponding K3 covers.

Let  $\Delta$  be the unit disk  $\{t \in \mathbb{C} \mid |t| < 1\}$  and let  $\Delta^* = \Delta \setminus \{0\}$ . We are interested in proper flat families  $\mathfrak{X}^* \rightarrow \Delta^*$  with  $\mathfrak{X}^*$  smooth and where  $\mathfrak{X}_t^*$  is a smooth K3 surface for all  $t \in \Delta^*$ . Equip the family  $\mathfrak{X}^*$  with an effective relative Cartier divisor  $\mathcal{H}^*$  such that  $(\mathfrak{X}^*, \mathcal{H}^*) \rightarrow \Delta^*$  is a family of stable pairs. Let  $\mathfrak{X}$  be a semistable degeneration with  $K_{\mathfrak{X}} \sim 0$  (or simply Kulikov degeneration for short) completing  $\mathfrak{X}^*$  over  $\Delta$  (see [Ku, PP]). Recall that the central fiber  $\mathfrak{X}_0$  can be of type I, II or III (see [Ku, Theorem II]). In type II, denote by  $j(\mathfrak{X}_0)$  the  $j$ -invariant of one of the mutually isomorphic genus 1 double curves on  $\mathfrak{X}_0$ . Then define  $\mathcal{H}$  to be the closure of  $\mathcal{H}^*$  inside  $\mathfrak{X}$  ( $\mathcal{H}$  is flat over  $\Delta$ ). We can define a second completion of  $(\mathfrak{X}^*, \mathcal{H}^*)$  over  $\Delta$ , which we denote by  $(\mathfrak{X}', \mathcal{H}')$ , such that  $(\mathfrak{X}'_0, \epsilon \mathcal{H}'_0)$  is a stable pair for all  $0 < \epsilon \ll 1$ .

**Definition 7.2.** With the notation introduced above, define the *dual graph* of  $\mathfrak{X}'_0$  as follows. Draw a vertex  $v_i$  for each irreducible component  $V_i$  of  $\mathfrak{X}'_0$ . Then, given any two distinct irreducible components  $V_i$  and  $V_j$ , draw one edge between  $v_i$  and  $v_j$  for each irreducible curve in  $V_i \cap V_j$ . If an irreducible component  $V_i$  self-intersects along a curve  $C$ , then draw one loop on  $v_i$  for each irreducible component of  $C$ . Denote by  $G(\mathfrak{X}'_0)$  the dual graph of  $\mathfrak{X}'_0$ .

**Definition 7.3.** Let  $\mathfrak{X}'$  as defined above. We say that  $\mathfrak{X}'_0$  has *generalized type I, II, or III* if the following hold:

- Type I:  $G(\mathfrak{X}'_0)$  consists of one vertex and  $\mathfrak{X}'_0$  has at worst Du Val singularities;
- Type II:  $G(\mathfrak{X}'_0)$  is a chain and  $\mathfrak{X}'_0$  has at worst elliptic singularities. If there are at least two vertices and the double curves are mutually isomorphic genus 1 curves, then denote by  $j(\mathfrak{X}'_0)$  the  $j$ -invariant of one of these;
- Type III: otherwise.

The proof of the following theorem, which builds upon the proof of [L16, Theorem 2.9], was communicated to me by Valery Alexeev.

**Theorem 7.4.** *With the notation introduced above,  $\mathfrak{X}_0$  and  $\mathfrak{X}'_0$  have the same type. In addition, if  $\mathfrak{X}_0, \mathfrak{X}'_0$  have type II and  $j(\mathfrak{X}'_0)$  can be defined, then  $j(\mathfrak{X}_0) = j(\mathfrak{X}'_0)$ .*

*Proof.* The proof of [L16, Theorem 2.9] describes a procedure to construct the unique stable model  $(\mathfrak{X}', \epsilon \mathcal{H}')$  by modifying  $(\mathfrak{X}, \mathcal{H})$ . This procedure consists mainly of the following two steps:

- Step 1: Replace  $\mathfrak{X}$  with another Kulikov degeneration such that  $\mathcal{H}$  is nef and it does not contain double curves or triple points (this may involve, among other things, base changes);
- Step 2: For  $n \geq 4$ , the line bundle  $\mathcal{O}_{\mathfrak{X}}(n\mathcal{H})$  induces a birational morphism  $(\mathfrak{X}, \mathcal{H}) \rightarrow (\mathfrak{X}', \mathcal{H}')$  where  $\mathfrak{X}' = \text{Proj}_{\Delta} \left( \bigoplus_{n \geq 0} \mathcal{O}_{\mathfrak{X}}(n\mathcal{H}) \right)$  (see [SB, Theorem 2, part (i)]). This birational morphism consists of the contraction of some components or curves in the central fiber  $\mathfrak{X}_0$ .

In step 1, the new Kulikov model is obtained from the one we started by applying elementary modifications of type 0, I and II (see [FM, pages 12–15] for their definitions), base changes, and blow ups of  $\mathfrak{X}_0$  along double curves or triple points. The elementary modifications and the blow ups obviously do not change the type of the central fiber. A description of how the central fiber is

modified after a base change can be found in [Fr], and also in this case the type does not change. In step 2, it follows from our definition of generalized type that  $\mathfrak{X}'_0$  has the same type as  $\mathfrak{X}_0$ .

This shows that if  $\mathfrak{X}_0$  has type I, II, or III, then  $\mathfrak{X}'_0$  has type I, II, or III respectively. The converse follows from this and from the uniqueness of the stable model. The claim about the  $j$ -invariants also follows from our discussion.  $\square$

**7.2. Map to the Baily-Borel compactification.** The period domain  $\mathcal{D}_L = \mathcal{D}$  parametrizing  $D_{1,6}$ -polarized Enriques surfaces (where  $L$  is the lattice  $\mathbb{Z}^2(2) \oplus \mathbb{Z}^4(-1)$ , see [O, Section 3.4.2] for the details) and its Baily-Borel compactification  $\overline{\mathcal{D}/\Gamma}^{BB}$  ( $\Gamma$  denotes the isometry group of  $L$ ) are studied in [O]. Oudompheng shows that the boundary of  $\overline{\mathcal{D}/\Gamma}^{BB}$  consists of two rational 1-cusps and three 0-cusps (see [O, Proposition 7.7]). The 1-cusps are labeled even and odd. The 0-cusps are labeled even, odd of type 1, and odd of type 2. These are arranged as shown in Figure 7. In what follows we construct a birational morphism  $\overline{M}_{D_{1,6}}^\nu \rightarrow \overline{\mathcal{D}/\Gamma}^{BB}$  which highlights a correspondence between the boundary strata of the two compactifications.

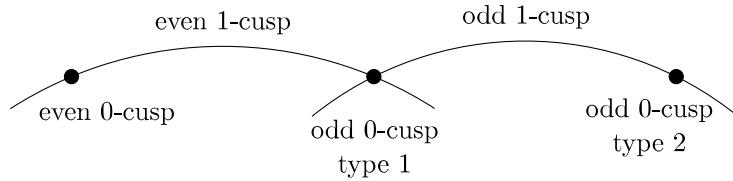


FIGURE 7. Boundary of the Baily-Borel compactification  $\overline{\mathcal{D}/\Gamma}^{BB}$

**Observation 7.5.** The two rational 1-cusps of the Baily-Borel compactification  $\overline{\mathcal{D}/\Gamma}^{BB}$  are degree 3 covers of the modular curve  $X(1) \cong \mathbb{P}^1$  (see [O, Section 7.3]).

**Lemma 7.6.** *There exists a compactification  $\overline{\mathcal{D}/\Gamma}'$  of  $\mathcal{D}/\Gamma$  obtained from  $\overline{\mathcal{D}/\Gamma}^{BB}$  by gluing the three 0-cusps together and by gluing the two 1-cusps to a rational curve with one node whose smooth points correspond to isomorphism classes of elliptic curves. Moreover,  $\overline{\mathcal{D}/\Gamma}^{BB}$  is the normalization of  $\overline{\mathcal{D}/\Gamma}'$ .*

*Proof.* The main technique we use for the proof is [Fe, Theorem 5.4]. Let us briefly recall it. Let  $X'$  be a scheme,  $Y'$  a closed subscheme of  $X'$ , and  $Y' \rightarrow Y$  a finite morphism. Consider the ringed space  $X = X' \amalg_{Y'} Y$  and the cocartesian square

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X. \end{array}$$

Let us assume that any finite sets of points in  $X'$  (resp.  $Y$ ) are contained in an open affine subset of  $X'$  (resp.  $Y$ ). Then  $X$  is a scheme verifying the same property on finite sets of points, the above diagram is cartesian,  $Y \rightarrow X$  is a closed immersion, the morphism  $X' \rightarrow X$  is finite, and it induces an isomorphism  $X' \setminus Y' \cong X \setminus Y$ .



Back to our case, denote by  $C_{\text{even}}, C_{\text{odd}}$  the two 1-cusps of  $\overline{\mathcal{D}}/\Gamma^{BB}$ . We want to first consider the finite morphism  $C_{\text{even}} \rightarrow X(1)$  to build the projective scheme  $X_1 = \overline{\mathcal{D}}/\Gamma^{BB} \amalg_{C_{\text{even}}} X(1)$  (the hypothesis on finite sets of points is satisfied because  $\overline{\mathcal{D}}/\Gamma^{BB}$  and  $X(1)$  are projective). Repeat the gluing on  $X_1$  by considering now  $C_{\text{odd}} \rightarrow X(1)$  to obtain  $X_2$ . Now glue together the two copies of  $X(1)$  on  $X_2$  to obtain  $X_3$ . Finally, identify the images in  $X_3$  of the three 0-cusps to obtain the claimed compactification  $\overline{\mathcal{D}}/\Gamma'$ . The isomorphism  $\overline{\mathcal{D}}/\Gamma^{BB} \cong (\overline{\mathcal{D}}/\Gamma')^\nu$  follows from the Zariski Main Theorem because  $\overline{\mathcal{D}}/\Gamma^{BB}$  is normal and the morphism  $\overline{\mathcal{D}}/\Gamma^{BB} \rightarrow \overline{\mathcal{D}}/\Gamma'$  is finite and birational.  $\square$

**Definition 7.7.** Recall the family  $\mathfrak{S}_{\text{sm}}/\mathcal{U}_{\text{sm}}$  in Definition 3.19. We denote by  $\mathfrak{S}_{\text{sm}}^{\text{K3}}$  the  $\mathbb{Z}_2^3$ -cover of  $\mathfrak{S}_{\text{sm}}$  branched along  $(\Delta \times \mathcal{U}_{\text{sm}})|_{\mathfrak{S}_{\text{sm}}}$ , which is a family of K3 surfaces. Observe that  $\mathfrak{S}_{\text{sm}}^{\text{K3}}$ , like  $\mathfrak{S}_{\text{sm}}$ , is embedded in  $(\mathbb{P}^1)^3 \times \mathcal{U}_{\text{sm}}$ .

**Theorem 7.8.** *There exists a birational morphism  $\overline{M}_{D_{1,6}}^\nu \rightarrow \overline{\mathcal{D}}/\Gamma^{BB}$  which maps the boundary of  $\overline{M}_{D_{1,6}}^\nu$  to the boundary of  $\overline{\mathcal{D}}/\Gamma^{BB}$ .*

*Proof.* The GIT interpretation of  $\overline{\mathcal{D}}/\Gamma^{BB}$  in [O] as quotient of the Grassmannian  $\text{Gr}(3, 6)$  gives us a birational morphism  $\overline{\mathcal{D}}/\Gamma^{BB} \dashrightarrow \overline{M}_{D_{1,6}}^\nu$  (this morphism is defined at points corresponding to arrangements of six lines in  $\mathbb{P}^2$  without triple intersection points). Given the open subset  $\mathcal{U}_{\text{sm}} \subset \overline{M}_Q^\nu$ , consider the composition  $\mathcal{U}_{\text{sm}} \rightarrow \overline{M}_{D_{1,6}}^\nu \dashrightarrow \overline{\mathcal{D}}/\Gamma^{BB}$  (which is regular). We show that this morphism extends to  $\overline{M}_Q^\nu$  giving a  $\text{Sym}(Q)$ -equivariant morphism. This extension is induced by the universal property of the normalization after we extend to  $\overline{M}_Q^\nu$  the composition  $\mathcal{U}_{\text{sm}} \rightarrow \overline{\mathcal{D}}/\Gamma^{BB} \rightarrow \overline{\mathcal{D}}/\Gamma'$ , which we denote by  $\rho$  (the compactification  $\overline{\mathcal{D}}/\Gamma'$  is constructed in Lemma 7.6). To extend  $\rho$  we use Theorem 6.2. So let  $K$  be the field of fractions of a DVR  $(A, \mathfrak{m})$  and consider any  $g: \text{Spec}(K) \rightarrow \mathcal{U}_{\text{sm}}$ . We show that  $\lim(\rho \circ g) \in \overline{\mathcal{D}}/\Gamma'$  can be computed using only  $\lim g \in \overline{M}_Q^\nu$ .

Let  $(X, B)$  be the stable toric pair parametrized by  $\lim g \in \overline{M}_Q^\nu$  and consider the corresponding stable pair  $(B^\bullet, (\frac{1+\epsilon}{2})\Delta^\bullet|_{B^\bullet})$ . We distinguish the following three cases:

- Case I:  $B^\bullet$  is irreducible;
- Case II:  $B^\bullet$  has exactly two irreducible components glued along an irreducible curve;
- Case III: otherwise.

Denote by  $p_0$  (resp.  $C_0$ ) the image of the 0-cusps (resp. 1-cusps) under the morphism  $\overline{\mathcal{D}}/\Gamma^{BB} \rightarrow \overline{\mathcal{D}}/\Gamma'$ . The point  $\lim(\rho \circ g)$  can be computed as follows.  $\mathcal{U}_{\text{sm}}$  carries the family of K3 surfaces  $(\mathfrak{S}_{\text{sm}}^{\text{K3}}, \epsilon(\Delta \times \mathcal{U}_{\text{sm}})|_{\mathfrak{S}_{\text{sm}}^{\text{K3}}})$ , where  $\Delta$  is the toric boundary of  $(\mathbb{P}^1)^3$ . Let  $\mathcal{Y}'$  be the completion over  $\text{Spec}(A)$  (or a finite ramified base change of it) of the restriction of  $\mathfrak{S}_{\text{sm}}^{\text{K3}}$  to  $\text{Spec}(K)$  (we omit the divisor for simplicity of notation). In particular,  $\mathcal{Y}'_{\mathfrak{m}}$  is the  $\mathbb{Z}_2^3$ -cover of  $B^\bullet$  branched along  $\Delta^\bullet|_{B^\bullet}$  and it is the stable model of a degeneration of smooth K3 surface pairs. Let  $\mathcal{Y}$  be a birational modification of  $\mathcal{Y}'$  which is a Kulikov degeneration. Then  $\mathcal{Y}_{\mathfrak{m}}$  determines a unique point in  $\overline{\mathcal{D}}/\Gamma'$ , which depends on the type of  $\mathcal{Y}_{\mathfrak{m}}$  as follows:

- Type I:  $\lim(\rho \circ g) \in \overline{\mathcal{D}}/\Gamma^{BB}$  is the image under the quotient  $\mathcal{D} \rightarrow \mathcal{D}/\Gamma$  of the period point corresponding to  $\mathcal{Y}_{\mathfrak{m}}$ ;
- Type II:  $\lim(\rho \circ g) \in C_0$  and it corresponds to  $j(\mathcal{Y}_{\mathfrak{m}})$ ;
- Type III:  $\lim(\rho \circ g) = p_0$ .

It is easy to observe that  $B^\bullet$  falls into case I (resp. II, III) if and only if  $\mathcal{Y}'_m$  has generalized type I (resp. II, III) (by Remark 5.8( $d1$ ), ( $d1'$ ), if  $B^\bullet$  falls into case I, then  $\mathcal{Y}'_m$  is smooth or it has isolated singularities of type  $A_1$  or  $A_3$ ). Moreover, the generalized type of  $\mathcal{Y}'_m$  equals the type of  $\mathcal{Y}_m$  by Theorem 7.4. In addition, if we are in case II, then it makes sense to define the  $j$ -invariant  $j(\mathcal{Y}'_m)$ , and this equals  $j(\mathcal{Y}_m)$  again by Theorem 7.4. In conclusion, we proved that  $\lim(\rho \circ g)$  only depends on  $\lim g$ .  $\square$

**Remark 7.9.** Consider the three 0-cusps of  $\overline{\mathcal{D}/\Gamma}^{BB}$  (even, odd of type 1, and odd of type 2). From the way we constructed the morphism  $\overline{M}_{D_{1,6}}^\nu \rightarrow \overline{\mathcal{D}/\Gamma}^{BB}$ , it is not clear to which one of these 0-cusps is a given 0-dimensional stratum mapped to. Let us briefly show that a 0-dimensional boundary stratum maps to the 0-cusp with the same label (recall the terminology for the 0-dimensional boundary strata in Theorem 6.7). It is enough to show that a given point in the interior of the maximal 1-dimensional stratum of  $\overline{M}_{D_{1,6}}^\nu$  is mapped to the 1-cusp  $C_{\text{odd}}$ . Consider a smooth one-parameter family with fibers isomorphic to  $\text{Bl}_3 \mathbb{P}^2$ . Equip this family with a divisor of coefficient  $\frac{1+\epsilon}{2}$  cutting on each fiber our usual configuration of three pairs of lines without triple points, exception made for the central fiber where two lines belonging to the same pair come together and the other four lines are general. It is easy to compute that the limit of this one-parameter family in  $\overline{M}_{D_{1,6}}^\nu$  lies on the maximal 1-dimensional stratum (to show this, one has to compute the stable replacement of the central fiber, but this is done by blowing up the double line and then contracting the strict transforms of the two  $(-1)$ -curves intersecting the double line). The limit of the same family in  $\overline{\mathcal{D}/\Gamma}^{BB}$  belongs to  $C_{\text{odd}}$ , and this can be deduced using the GIT interpretation of  $\overline{\mathcal{D}/\Gamma}^{BB}$  in terms of degenerations of line arrangements in  $\text{Bl}_3 \mathbb{P}^2$  (see [O]). This shows what we claimed.

**7.3. Comparison between the boundaries of  $\overline{M}_{D_{1,6}}^\nu$  and  $\overline{\mathcal{D}/\Gamma}^{BB}$ .** We want to analyze the preimages of the three 0-cusps of  $\overline{\mathcal{D}/\Gamma}^{BB}$  under the morphism  $\overline{M}_{D_{1,6}}^\nu \rightarrow \overline{\mathcal{D}/\Gamma}^{BB}$  of Theorem 7.8.

**Observation 7.10.** It follows from the proof of Theorem 7.8 and from Remark 7.9 that the preimage of the even 0-cusp (resp. odd of type 1, odd of type 2) consists of the union of the strata of  $\overline{M}_{D_{1,6}}^\nu$  containing the even 0-dimensional stratum (resp. odd of type 1, odd of type 2), and parametrizing degenerate stable pairs whose  $\mathbb{Z}_2^3$ -cove has generalized type III (see Definition 7.3). In what follows, we refer to the stable pairs  $(a), (b), (c2), (c3), (d3)$  in Figure 3. Observe that  $(b), (c2)$  have 1-dimensional moduli, and  $(a), (c3), (d3)$  are unique up to isomorphism.

**7.3.1. Even 0-cusp.** The preimage of the even 0-cusp is the union of the boundary strata whose points parametrize stable pairs obtained by gluing copies of  $(a), (b)$  as shown in Figure 6. In particular, this preimage consists of the union of a divisor and a codimension 2 boundary stratum. In Section 8 we show that the boundary of  $\overline{M}_{D_{1,6}}^\nu$  has a toroidal behavior in a neighborhood of the preimage of the even 0-cusp.

**7.3.2. Odd 0-cusp of type 1.** The preimage of the odd 0-cusp of type 1 is a codimension 2 stratum. A point in this stratum parametrizes one of the following stable pairs:

- Glue two  $(c2)$ ;
- Glue  $(c2)$  and  $(c3)$ ;
- Glue two  $(c3)$  (this is the stable pair parametrized by the odd 0-stratum of type 1).

The behavior of the compactification  $\overline{M}_{D_{1,6}}^\nu$  in a neighborhood of the preimage of this 0-cusp is neither toroidal nor Baily-Borel (more about this is discussed in Section 8).

**7.3.3. Odd 0-cusp of type 2.** By the stratification of the boundary of  $\overline{M}_{D_{1,6}}^\nu$  in Figure 6, we can see that the preimage of the odd 0-cusp of type 2 consists only of the point parametrizing the stable pair  $(d3)$ .

**Observation 7.11.** The compactification  $\overline{M}_{D_{1,6}}^\nu$  is isomorphic to the Baily-Borel compactification in a neighborhood of the preimage of this 0-cusp. To prove this isomorphism, just observe that the restriction of the morphism  $\overline{M}_{D_{1,6}}^\nu \rightarrow \overline{\mathcal{D}}/\Gamma^{BB}$  to this neighborhood is birational and finite, hence an isomorphism by the Zariski Main Theorem because  $\overline{\mathcal{D}}/\Gamma^{BB}$  is normal.

## 8. RELATION WITH LOOIJENGA'S SEMITORIC COMPACTIFICATIONS

The behavior of the boundary of  $\overline{M}_{D_{1,6}}^\nu$  suggests that  $\overline{M}_{D_{1,6}}^\nu$  can possibly be isomorphic to a semitoric compactification  $\overline{\mathcal{D}}/\Gamma^\Sigma$ . These semitoric compactifications were introduced in [Lo], and they depend on the choice of  $\Sigma$ , which is called an *admissible decomposition of the conical locus of  $\mathcal{D}$*  (see Definition 8.3). In what follows, we provide a natural choice of  $\Sigma$  such that the boundary strata of the corresponding semitoric compactification  $\overline{\mathcal{D}}/\Gamma^\Sigma$  are in bijection with the boundary strata of  $\overline{M}_{D_{1,6}}^\nu$ . These considerations make us conjecture that  $\overline{\mathcal{D}}/\Gamma^\Sigma$  is actually isomorphic to  $\overline{M}_{D_{1,6}}^\nu$ . We will address this conjecture in future work.

### 8.1. Admissible decomposition of the conical locus.

**Notation 8.1.** We adopt the same notation as in [Lo]. Let  $V = L \otimes_{\mathbb{Z}} \mathbb{C}$ , where the lattice  $L = \mathbb{Z}^2(2) \oplus \mathbb{Z}^4(-1)$  was introduced in Section 7.2. We call  $\mathbb{Q}$ -isotropic an isotropic subspace  $W \subset V$  which is defined over  $\mathbb{Q}$ . Denote by  $I$  (resp.  $J$ ) a  $\mathbb{Q}$ -isotropic line (resp. plane) in  $V$ . Denote by  $C_I \subset (I^\perp/I)(\mathbb{R})$  (resp.  $C_J \subset \bigwedge^2 J(\mathbb{R})$ ) one of the two connected components of  $\{x \in (I^\perp/I)(\mathbb{R}) \mid x \cdot x > 0\}$  (resp.  $\bigwedge^2 J(\mathbb{R}) \setminus \{0\}$ ). There is a canonical choice for  $C_I$  and  $C_J$  if we specify a connected component of  $\mathcal{D}$ , as it is explained in [Lo, Sections 1.1 and 1.2] (in this case, we let  $\Gamma$  be the subgroup of isometries of  $L$  which preserve the chosen connected component). Denote by  $C_{I,+}$  (resp.  $C_{J,+}$ ) the convex hull of the  $\mathbb{Q}$ -vectors in  $\overline{C}_I$  (resp.  $\overline{C}_J$ ).

**Definition 8.2** ([Lo, Definition 2.1]). The *conical locus* of  $\mathcal{D}$  is defined as

$$C(\mathcal{D}) = \coprod_{\substack{W \subset V, \\ W \text{ } \mathbb{Q}\text{-isotropic}}} C_W.$$

**Definition 8.3** ([Lo, Definition 6.1]). An *admissible decomposition* of  $C(\mathcal{D})$  is a  $\Gamma$ -invariant locally rational decomposition of  $C_{I,+}$  for all  $\mathbb{Q}$ -isotropic lines  $I$  such that, for all  $\mathbb{Q}$ -isotropic planes  $J$ , the support space of  $C_{J,+} \subset (I^\perp/I)(\mathbb{R})$  (see [Lo, Lemma-Definition 4.5]) is independent of  $I$  when we regard that support space as a subspace of  $J^\perp$  containing  $J$ .

We are ready to define our choice of admissible decomposition of  $C(\mathcal{D})$ .

**Definition 8.4.** For any  $\mathbb{Q}$ -isotropic line  $I$ , consider the decomposition of  $C_{I,+}$  induced by the mirrors of the reflections with respect to the vectors of square  $-1$  in the hyperbolic lattice  $(I^\perp/I)(\mathbb{Z})$  (we call these vectors  $(-1)$ -vectors). Let  $\Sigma$  be the decomposition of  $C(\mathcal{D})$  induced by these mirrors. This is an admissible decomposition of  $C(\mathcal{D})$ , as we show in Corollary 8.10 after the considerations in Section 8.2.

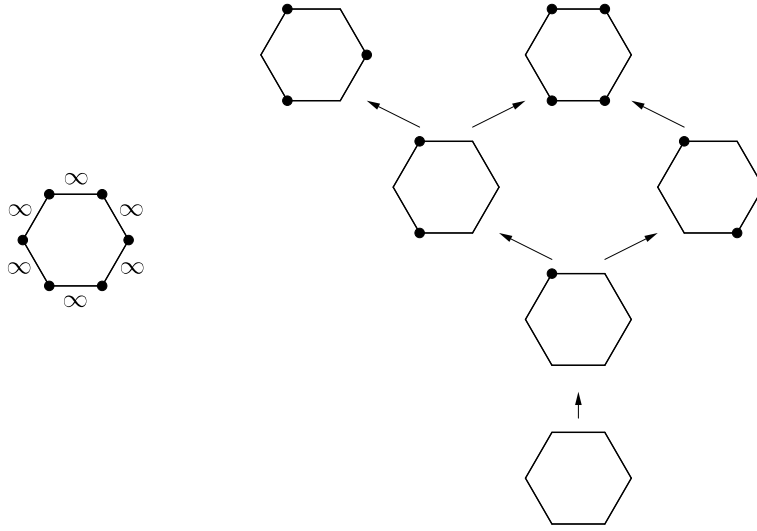


FIGURE 8. On the left, we show the Coxeter diagram for the discrete reflection group  $\Gamma_I$ , where  $I$  corresponds to the even 0-cusp. On the right are its elliptic and maximal parabolic subdiagrams, up to isometries

**8.2. Study of the decomposition  $\Sigma$ .** There are three choices of  $\mathbb{Q}$ -isotropic lines  $I \subset V$  up to isometries of  $L$ . In each case, the hyperbolic lattice  $(I^\perp/I)(\mathbb{Z})$  is computed in [O, Proposition 7.5]. Let  $\Gamma_I$  be the discrete reflection group generated by the reflections with respect to the  $(-1)$ -vectors in  $(I^\perp/I)(\mathbb{Z})$ .

**8.2.1. Even 0-cusp.** For  $I$  corresponding to the even 0-cusp, we have that  $(I^\perp/I)(\mathbb{Z}) \cong \mathbb{Z}^{1,3}$ , which denotes the unique odd unimodular lattice of signature  $(1,3)$ . Running Vinberg's algorithm to determine a fundamental domain for  $\Gamma_I$  (see [Vin, Section 1] and [Vin, Theorem 2.6 bis] for the stopping condition), we obtain the *Coxeter diagram* (see [Vin, Section 1] or [Sc, Table 2.2.8]) shown on the left in Figure 8. For the computations, we chose  $(1, 0, 0, 0)$  as our initial vector.

**Observation 8.5.** There is a bijection between the strata of the boundary of  $\overline{M}_{D_{1,6}}^\nu$  containing the even 0-dimensional stratum (see Figure 6) and the elliptic and maximal parabolic subdiagrams of the Coxeter diagram up to isometries of  $\mathbb{Z}^{1,3}$  (on the right in Figure 8).

**Remark 8.6.** Let  $J$  be a  $\mathbb{Q}$ -isotropic plane containing  $I$ . Consider the hyperplanes  $v^\perp \subset (I^\perp/I)(\mathbb{R})$  where  $v \in (I^\perp/I)(\mathbb{Z})$  is a  $(-1)$ -vector and  $C_{J,+} \subset v^\perp$ . All these hyperplanes intersect  $C_I$  nontrivially and their intersection is  $C_{J,+}$ . Therefore, the support space of  $C_{J,+}$  is equal to  $J$  when we regard it as a subspace of  $J^\perp$  containing  $J$ .

**8.2.2. Odd 0-cusp of type 1.** If  $I$  corresponds to the odd 0-cusp of type 1, then  $(I^\perp/I)(\mathbb{Z}) \cong \mathbb{Z}^{1,1} \oplus \mathbb{Z}^2(-2)$ . We run Vinberg's algorithm to determine a fundamental domain for  $\Gamma_I$ . Choose  $x_0 = (1, -1, 0, 0)$  as initial vector. In step 0 of the algorithm, we first determine all the  $(-1)$ -vectors which are orthogonal to  $v_0$ , and there are no such vectors. In step 1, we determine all the  $(-1)$ -vectors  $x$  such that  $v_0 \cdot x = 1$ . These vectors are given by

$$\alpha_{(a,b)} = (a^2 + b^2, 1 - a^2 - b^2, a, b), \text{ with } (a, b) \in \mathbb{Z}^2.$$

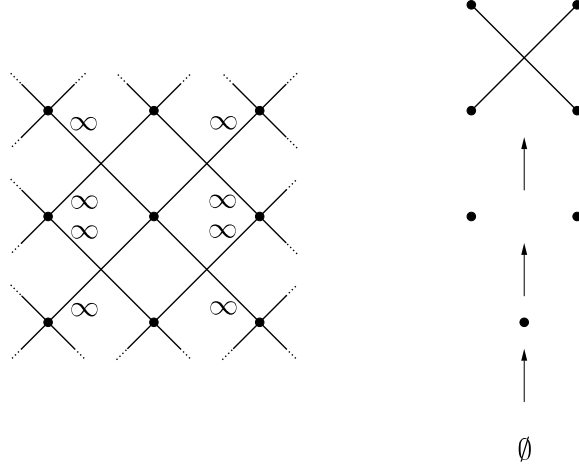


FIGURE 9. On the left, we show part of the Coxeter diagram for  $\Gamma_I$  where we omit the dotted edges. Here  $I$  corresponds to the odd 0-cusp of type 1. On the right are its elliptic and maximal parabolic subdiagrams, up to isometries

At step  $n \geq 2$  no new vectors are added. More precisely, if  $n$  is even, then there are no  $(-1)$ -vectors such that  $v_0 \cdot x = n$ . If  $n$  is odd, let  $x$  be a  $(-1)$ -vector such that  $v_0 \cdot x = n$ . Then  $x$  is not accepted by the algorithm because it is easy to check that  $x \cdot \alpha_{(a,b)} < 0$  if we choose  $(a, b) \in \mathbb{Z}^2$  such that  $|a - \frac{x_3}{n}|, |b - \frac{x_4}{n}| \leq \frac{1}{2}$ . Therefore,  $\mathbb{Z}^2$  is the set of vertices of the Coxeter diagram for  $\Gamma_I$ , where  $(a, b) \in \mathbb{Z}^2$  corresponds to  $\alpha_{(a,b)}$ . Observe that

$$\alpha_{(a,b)} \cdot \alpha_{(c,d)} = -1 + (a - c)^2 + (b - d)^2.$$

This implies that, given any  $(a, b) \in \mathbb{Z}^2$ , then

- There is no edge between  $(a, b)$  and  $(a \pm 1, b), (a, b \pm 1)$ ;
- There is an edge labeled by  $\infty$  between  $(a, b)$  and  $(a \pm 1, b \pm 1), (a \pm 1, b \mp 1)$ ;
- $(a, b)$  is connected by a dotted edge to any other vertex different from  $(a \pm 1, b), (a, b \pm 1), (a \pm 1, b \pm 1), (a \pm 1, b \mp 1)$ .

Part of the infinite Coxeter diagram is shown on the left in Figure 9. (The computation we just carried out shares some similarities with [Con].)

**Observation 8.7.** Also in this case there is a bijection between the strata of the boundary of  $\overline{M}_{D_{1,6}}^\nu$  containing the odd 0-dimensional stratum of type 1 (see Figure 6) and the elliptic and maximal parabolic subdiagrams of the Coxeter diagram up to isometries of  $\mathbb{Z}^{1,1} \oplus \mathbb{Z}^2(-2)$  (on the right in Figure 9).

**Remark 8.8.** Let  $J$  be a  $\mathbb{Q}$ -isotropic plane containing  $I$  and consider  $C_{J,+}$ . Up to isometries there are two possibilities for  $C_{J,+}$ . Assume that  $C_{J,+}$  is the ray generated by  $(2, 0, 1, 1)$  in  $C_{I,+} \subset (I^\perp/I)(\mathbb{R})$ . One can easily see that the support space of  $C_{J,+}$  is  $J$  (this is similar to what we did in Remark 8.6). If  $C_{J,+}$  is the ray generated by  $(1, -1, 0, 0)$ , then  $C_{J,+}$  is not orthogonal to any  $(-1)$ -vector in  $(I^\perp/I)(\mathbb{Z})$ , hence its support space is  $J^\perp$ . By [O, Lemma 7.2], we have that  $J$  corresponds to the even 1-cusp in the first case, and the odd 1-cusp in the second case.

8.2.3. *Odd 0-cusp of type 2.* Let  $H$  be the unique even unimodular lattice of signature  $(1, 1)$ . Then for  $I$  corresponding to the odd 0-cusp of type 2, we have that  $(I^\perp/I)(\mathbb{Z}) \cong H \oplus \mathbb{Z}^2(-2)$ , which is an even lattice. In particular,  $(I^\perp/I)(\mathbb{Z})$  does not contain  $(-1)$ -vectors. Therefore, the decomposition of  $C_{I,+}$  induced by  $\Sigma$  is given by  $C_{I,+}$  itself.

**Remark 8.9.** If  $J$  is a  $\mathbb{Q}$ -isotropic plane containing  $I$ , then the support space of  $C_{J,+}$  is  $J^\perp$ .

### 8.3. Conclusions.

**Corollary 8.10.** *The decomposition  $\Sigma$  in Definition 8.4 is an admissible decomposition of  $C(\mathcal{D})$ , and hence it gives rise to a semitoric compactification  $\overline{\mathcal{D}/\Gamma}^\Sigma$ .*

*Proof.* We have that  $\Sigma$  is admissible because Remarks 8.6, 8.8, and 8.9 imply that, for any  $\mathbb{Q}$ -isotropic plane  $J$ , the support space of  $C_{J,+}$  is independent from the choice of  $\mathbb{Q}$ -isotropic line  $I \subset J$ .  $\square$

**Observation 8.11.** The Baily-Borel compactification  $\overline{\mathcal{D}/\Gamma}^{BB}$  can be thought of as the semitoric compactification of  $\mathcal{D}/\Gamma$  associated to the trivial admissible decomposition of  $C(\mathcal{D})$  (see [Lo, Example 6.2]). In particular, by [Lo, Lemma 6.6] we have a birational morphism  $\overline{\mathcal{D}/\Gamma}^\Sigma \rightarrow \overline{\mathcal{D}/\Gamma}^{BB}$  which is an isomorphism on  $\mathcal{D}/\Gamma$ .

**Theorem 8.12.** *Consider the two birational modifications  $\overline{M}_{D_{1,6}}^\nu \rightarrow \overline{\mathcal{D}/\Gamma}^{BB} \leftarrow \overline{\mathcal{D}/\Gamma}^\Sigma$  of the Baily-Borel compactification of  $\mathcal{D}/\Gamma$ . Then these are isomorphic in a neighborhood of the preimage of the odd 1-cusp of type 2. Moreover, there is an intersection-preserving bijection between the boundary strata of the two compactifications which also preserves the dimensions of the strata.*

*Proof.* The statement of the theorem summarizes Observations 7.11, 8.5, and 8.7.  $\square$

**Conjecture 8.13.** *The KSBA compactification  $\overline{M}_{D_{1,6}}^\nu$  and the semitoric compactification  $\overline{\mathcal{D}/\Gamma}^\Sigma$  of  $\mathcal{D}/\Gamma$  are isomorphic.*

**Remark 8.14.** To prove Conjecture 8.13 the first step would be to show that the birational map  $\overline{M}_{D_{1,6}}^\nu \dashrightarrow \overline{\mathcal{D}/\Gamma}^\Sigma$  is an isomorphism in a neighborhood of the preimage of the even 0-cusp, which is where we have a toroidal behavior. To prove this, we plan to use the standard extension criterion used for instance in [AB, CMGHL], but adapted to our specific situation. For this adaptation, the results in [L08] will be helpful to us.

## 9. FINAL REMARKS

**Remark 9.1.** In the case considered in this paper, it is remarkable that the theory of stable pairs produced a compactification  $\overline{M}_{D_{1,6}}^\nu$  which has a toroidal behavior at the even 0-dimensional stratum, is isomorphic to the Baily-Borel compactification at the odd 0-dimensional stratum of type 2, and is a “mixture” of toroidal and Baily-Borel at the odd 0-dimensional stratum of type 1. In particular, this illustrates the behavior that should be expected in general when using stable pairs to compactify moduli spaces.

**Remark 9.2.** Associated to our 4-dimensional family of  $D_{1,6}$ -polarized Enriques surfaces there is a 4-dimensional family of K3 surfaces given by the  $\mathbb{Z}_2^3$ -covers  $X \rightarrow \text{Bl}_3 \mathbb{P}^2$  branched along three pairs of lines (see also Remark 3.15). One can also consider the minimal desingularization of the  $\mathbb{Z}_2$ -cover of  $\mathbb{P}^2$  branched along six lines, which gives rise to a second 4-dimensional family

of K3 surfaces. These two families are distinct and they are related as follows. We know from Remark 3.15 that  $X$  can be viewed as a  $(2, 2, 2)$  hypersurface in  $(\mathbb{P}^1)^3$ . The group action  $\mathbb{Z}_2^3 \curvearrowright X$  can be realized as the action of the restriction to  $X$  of the involutions of  $(\mathbb{P}^1)^3$  given in an affine patch by  $(x, y, z) \mapsto ((-1)^i x, (-1)^j y, (-1)^k z)$ ,  $i, j, k \in \{0, 1\}$ . The subgroup  $G < \mathbb{Z}_2^3$  corresponding to  $i + j + k = 2$  is isomorphic to  $\mathbb{Z}_2^2$  and acts symplectically on  $X$ . This means that the minimal resolution of  $X/G$  is again a K3 surface, which in this case belongs to the second 4-dimensional family of K3 surfaces.

**Remark 9.3.** Degenerations of six undistinguished lines in  $\mathbb{P}^2$  with weight  $\frac{1}{2} + \epsilon$  were considered in [A15, Section 6.2.1]. However, our case is different for the following reasons:

- First of all, we already explained in Remark 3.12 that our situation cannot be reduced to considering line arrangements in  $\mathbb{P}^2$ ;
- We keep track of three pairs of lines, and this is necessary to reconstruct the overlying degenerations of Enriques surfaces;
- Given a degeneration of  $\mathbb{P}^2$  with the six lines, it is not clear how to immediately obtain the corresponding degeneration of  $\text{Bl}_3 \mathbb{P}^2$  and the divisor on it;
- The boundaries of the two compactifications parametrizing in the interior six lines in  $\mathbb{P}^2$  and three pairs of lines in  $\text{Bl}_3 \mathbb{P}^2$  respectively are distinct, and the reason is the following. A  $D_{1,6}$ -polarized Enriques surface  $S$  can be realized as the minimal desingularization of the  $\mathbb{Z}_2^2$ -cover of  $\mathbb{P}^2$  branched along three pairs of lines. In this way  $S$  comes with a natural polarization of degree 4, i.e. the pullback of a line in  $\mathbb{P}^2$ . However, the polarization coming from the  $\mathbb{Z}_2^2$ -cover of  $\text{Bl}_3 \mathbb{P}^2$  has degree 6 and is given by  $E_1 + E_2 + E_3$  (see Observation 3.4).

## 10. APPENDIX

### 10.1. Completion of the proof of Proposition 3.16.

**Theorem 10.1.** *Let  $Q$  be the unit cube and let  $\mathcal{P}$  be a polyhedral subdivision of  $(Q, Q \cap \mathbb{Z}^3)$ . Then  $H^1(\mathcal{P}, \underline{\mathbb{T}}) = \{1\}$  (see Remark 2.2).*

To prove Theorem 10.1 we need some general preliminaries.

**Observation 10.2.** Let  $\mathcal{P}$  be a complex of equidimensional polytopes together with a reference map  $\cup_{P \in \mathcal{P}} P \rightarrow M_{\mathbb{R}}$ . Let  $\{P_i\}_{i \in I}$  be the set of maximal dimensional polytopes in  $\mathcal{P}$  and recall that  $\Lambda_{P_i}$  (which we denote for brevity  $\Lambda_i$ ) is the saturated sublattice of  $\mathbb{Z} \oplus M$  generated by  $(1, P_i)$ . For any  $\{i_1, \dots, i_s\} \subset I$ , let  $\Lambda_{i_1 \dots i_s}$  be the intersection  $\Lambda_{i_1} \cap \dots \cap \Lambda_{i_s}$ . An element of  $H^1(\mathcal{P}, \underline{\mathbb{T}})$  is a tuple  $(\dots, t_{ij}, \dots)$  indexed by pairs in  $I$ , with  $t_{ij} \in \text{Hom}(\Lambda_{ij}, \mathbb{k}^*)$  satisfying the *cocycle condition* and *modulo coboundaries*. The cocycle condition is the following: if  $i, j, k \in I$  is any triple, then we must have

$$t_{ij}|_{\Lambda_{ijk}} t_{jk}|_{\Lambda_{ijk}} = t_{ik}|_{\Lambda_{ijk}}.$$

A coboundary is a tuple  $(\dots, t_i|_{\Lambda_{ij}} t_j|_{\Lambda_{ij}}^{-1}, \dots)$  for  $t_i \in \text{Hom}(\Lambda_i, \mathbb{k}^*)$  and  $t_j \in \text{Hom}(\Lambda_j, \mathbb{k}^*)$ . Therefore, we have an equivalence relation on the cocycles given by

$$(\dots, t_{ij}, \dots) \sim (\dots, t_{ij} t_i|_{\Lambda_{ij}}^{-1} t_j|_{\Lambda_{ij}}, \dots).$$

**Lemma 10.3.** *Let  $\mathcal{P}$  be a connected complex of polytopes obtained by gluing finitely many polytopes of the same dimension. Let  $P$  be one of these polytopes and assume  $P$  has exactly one facet in common with some other maximal dimensional polytope in  $\mathcal{P}$ . Let  $\mathcal{P}'$  be the complex of polytopes obtained from  $\mathcal{P}$  by eliminating  $P$ . Then  $H^1(\mathcal{P}, \underline{\mathbb{T}}) \cong H^1(\mathcal{P}', \underline{\mathbb{T}})$ .*

*Proof.* Label  $P$  by 1 and the polytope adjacent to  $P$  by 2. Consider an arbitrary cocycle  $c = (t_{12}, \dots)$ . Let  $t_1 \in \text{Hom}(\Lambda_1, \mathbb{k}^*)$  be such that  $t_1|_{\Lambda_{12}} = t_{12}$  (observe that there can be different choices for  $t_1$ ). If we consider the coboundary  $(t_1|_{\Lambda_{12}} 1|_{\Lambda_{12}}^{-1}, 1, \dots)$ , then  $c \sim (1, \dots)$ , giving the required isomorphism.  $\square$

**Definition 10.4.** We call a polytope  $P$  as in the statement of Lemma 10.3 a *hanging* polytope of  $\mathcal{P}$ .

**Lemma 10.5.** Let  $\mathcal{P}$  be a connected complex of polytopes obtained by gluing finitely many polytopes  $\{P_i\}_{i=1}^n$  of the same dimension. Assume that all these polytopes share a fixed face of codimension 2 and that they are organized around it as follows: for  $i = 1, \dots, n-1$ ,  $P_i$  and  $P_{i+1}$  share a facet, and also  $P_1$  and  $P_n$  do. Then  $H^1(\mathcal{P}, \underline{T}) = \{1\}$ .

*Proof.* Let  $c = (t_{12}, t_{23}, \dots, t_{n-1,n}, t_{n1})$  be a cocycle and consider a coboundary with  $t_n|_{\Lambda_{n1}} = t_{n1}$  and the other  $t_i$  equal to 1. Then  $c \sim (t'_{12}, t'_{23}, \dots, t'_{n-1,n}, 1)$ . We can iterate this strategy until we obtain  $c \sim (\tau, 1, \dots, 1)$  for some  $\tau \in \text{Hom}(\Lambda_{12}, \mathbb{k}^*)$ . Now consider a coboundary with  $t_1|_{\Lambda_{12}} = \tau$ ,  $t_1|_{\Lambda_{n1}} = 1$  and the other  $t_i$  equal to 1. This makes sense because  $\tau|_{\Lambda_{n12}} = 1$  by the cocycle condition on  $(\tau, 1, \dots, 1)$ . We can conclude that  $c \sim (\tau, 1, \dots, 1) \sim (1, \dots, 1)$ .  $\square$

*proof of Theorem 10.1.* Let  $\mathcal{P}$  be any polyhedral subdivision of  $(Q, Q \cap \mathbb{Z}^3)$  where  $Q$  is the unit cube. Applying Lemma 10.3, we can reduce our problem to computing  $H^1(\mathcal{P}', \underline{T})$  with  $\mathcal{P}'$  without hanging polytopes (see Definition 10.4). After enumerating all the possibilities for  $\mathcal{P}$ , one can observe that either  $\mathcal{P}'$  is generated by a single maximal dimensional polytope, or it is as in the statement of Lemma 10.5. In both cases, we can conclude that  $H^1(\mathcal{P}, \underline{T}) \cong H^1(\mathcal{P}', \underline{T}) \cong \{1\}$ .

A complete list of all the possibilities for  $\mathcal{P}$  can be recovered from the table in Figure 10. A polyhedral subdivision of  $(Q, Q \cap \mathbb{Z}^3)$  is represented by a graph whose vertices correspond to the maximal dimensional polytopes in the subdivision and there is an edge between two vertices if and only if the corresponding polytopes share a facet. When this graph is not enough to distinguish two distinct polyhedral subdivisions, we associate a number to each vertex which represents the number of facets of the corresponding polytope. If this is not enough, instead we draw a dotted edge between two vertices if the corresponding polytopes share an edge and not a facet. The sum in the top left corner of each box in Figure 10 represents the lattice volume of the polytopes used in the subdivision.  $\square$

## 10.2. Completion of the proof of Theorem 5.3.

**Proposition 10.6.** Given  $P$  of type (c), then the pair  $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$  is stable (see Definition 5.4 for the definition of type (c)).

*Proof.*

$$\begin{aligned} X_P^\bullet &= \mathbb{P}^2 \times \mathbb{P}^1 \ni ([X_0 : X_1 : X_2], [Y_0 : Y_1]), \\ D_P^\bullet &= V(X_2), \\ \Delta_P^\bullet - D_P^\bullet &= V(X_0 X_1 Y_0 Y_1), \\ B_P^\bullet &= V((a_0 X_0 + a_1 X_1 + a_2 X_2) Y_0 + (b_0 X_0 + b_1 X_1 + b_2 X_2) Y_1). \end{aligned}$$

where  $a_0, a_1, b_0, b_1 \neq 0$  and exactly one among  $a_2$  and  $b_2$  can possibly be zero (there is at most one corner cut contained in  $P$ ). Let us start by assuming that  $a_2 b_2 \neq 0$ .





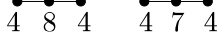






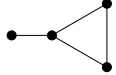
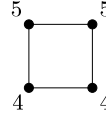
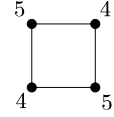


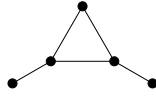
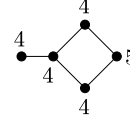
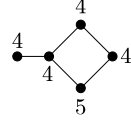
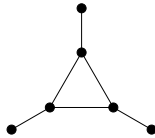
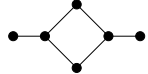
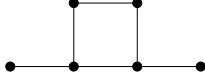
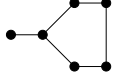
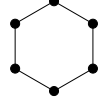
6	5 + 1	4 + 1 + 1	4 + 2
			No subdivisions
3 + 3	3 + 2 + 1	3 + 1 + 1 + 1	2 + 2 + 2
			
2 + 2 + 1 + 1	    		
2 + 1 + 1 + 1 + 1	    		
1 + 1 + 1 + 1 + 1 + 1	    		

FIGURE 10. List of all the polyhedral subdivisions of  $(Q, Q \cap \mathbb{Z}^3)$ ,  $Q = [0, 1]^3$ 

If  $B_P^\bullet$  is singular, assume without loss of generality that there is a singular point in the affine patch where  $X_0 \neq 0$  and  $Y_0 \neq 0$ . Let  $x_i = \frac{X_i}{X_0}$ ,  $i = 1, 2$ , and  $y = \frac{Y_1}{Y_0}$  be the coordinates in this affine patch. Then the equation of  $B_P^\bullet$  becomes

$$a_0 + a_1x_1 + a_2x_2 + (b_0 + b_1x_1 + b_2x_2)y = 0.$$

Therefore the following system of equations has a solution:

$$\begin{cases} a_0 + a_1x_1 + a_2x_2 + (b_0 + b_1x_1 + b_2x_2)y = 0, \\ a_1 + b_1y = 0, \\ a_2 + b_2y = 0, \\ b_0 + b_1x_1 + b_2x_2 = 0, \end{cases} \Rightarrow \begin{cases} y = -\frac{a_1}{b_1}, \\ y = -\frac{a_2}{b_2}, \\ a_1x_1 + a_2x_2 = -a_0, \\ b_1x_1 + b_2x_2 = -b_0. \end{cases}$$

This implies that  $a_1b_2 - a_2b_1 = 0$ , and therefore the two vectors  $(a_0, a_1, a_2)$  and  $(b_0, b_1, b_2)$  are proportional because the matrix  $\begin{pmatrix} a_1 & a_2 & -a_0 \\ b_1 & b_2 & -b_0 \end{pmatrix}$  has rank 1. If  $(b_0, b_1, b_2) = \lambda(a_0, a_1, a_2)$  for  $\lambda \in \mathbb{k}^*$ , then the equation of  $B_P^\bullet$  becomes

$$(a_0X_0 + a_1X_1 + a_2X_2)(Y_0 + \lambda Y_1) = 0,$$

where  $V(Y_0 + \lambda Y_1) \cong \mathbb{P}^2$  and  $V(a_0X_0 + a_1X_1 + a_2X_2) \cong \mathbb{P}^1 \times \mathbb{P}^1$  are glued along a ruling of  $\mathbb{P}^1 \times \mathbb{P}^1$  and the line  $a_0X_0 + a_1X_1 + a_2X_2 = 0$  in  $\mathbb{P}^2$ . The restrictions of  $D_P^\bullet$  and  $\Delta_P^\bullet - D_P^\bullet$  to these two irreducible components are described in Remark 5.8(c3). In this case, to conclude

that  $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$  is stable, we reduce the question to each irreducible component of  $B_P^\bullet$  and we apply what we already proved in the (a), (b) cases (see the proofs of Propositions 5.5 and 5.7).

Now let us assume that  $B_P^\bullet$  is smooth (and hence irreducible). By the discussion above, the two vectors  $(a_0, a_1, a_2)$  and  $(b_0, b_1, b_2)$  are not proportional. Denote by  $p$  the point of intersection of the two lines  $a_0X_0 + a_1X_1 + a_2X_2 = 0$  and  $b_0X_0 + b_1X_1 + b_2X_2 = 0$  in  $\mathbb{P}^2$ . If  $\pi: B_P^\bullet \rightarrow \mathbb{P}^2$  is the restriction to  $B_P^\bullet$  of the usual projection map  $\mathbb{P}^2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$ , observe that  $\pi$  is an isomorphism if restricted to the complement of  $\pi^{-1}(p)$  and  $\pi^{-1}(p) \cong \mathbb{P}^1$ . This proves that  $B_P^\bullet \cong \mathbb{F}_1$ . In this case, let us explain how  $D_P^\bullet|_{B_P^\bullet}$  depends on the coefficients  $a_i, b_j$ . The restriction  $D_P^\bullet|_{B_P^\bullet}$  has equation

$$\begin{cases} X_2 = 0, \\ (a_0X_0 + a_1X_1)Y_0 + (b_0X_0 + b_1X_1)Y_1 = 0. \end{cases}$$

By using an argument completely analogous to what we just did for  $B_P^\bullet$ , we have that this restriction is irreducible if and only if  $(a_0, a_1)$  and  $(b_0, b_1)$  are not proportional. In this case,  $D_P^\bullet|_{B_P^\bullet}$  is a section of  $B_P^\bullet$  with self-intersection 1. If  $(a_0, a_1)$  and  $(b_0, b_1)$  are proportional, then the equation of  $D_P^\bullet|_{B_P^\bullet}$  becomes

$$\begin{cases} X_2 = 0, \\ (a_0X_0 + a_1X_1)(Y_0 + \lambda Y_1) = 0. \end{cases}$$

for some  $\lambda \in \mathbb{k}^*$ . The irreducible component  $V(X_2, a_0X_0 + a_1X_1)$  (resp.  $V(X_2, Y_0 + \lambda Y_1)$ ) is the exceptional section (resp. a fiber) of  $B_P^\bullet$ . We are left with understanding  $(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}$ , and for this we need to study how  $V(X_i), V(Y_i), i = 0, 1$ , restrict to  $B_P^\bullet$ . But  $V(Y_i)$  restricts giving a fiber, and we can study  $V(X_i)|_{B_P^\bullet}$  in the same way we did for  $D_P^\bullet|_{B_P^\bullet}$ . Observe that at most one among  $V(X_0)|_{B_P^\bullet}, V(X_1)|_{B_P^\bullet}, D_P^\bullet|_{B_P^\bullet}$  can be reducible (otherwise  $B_P^\bullet$  would be reducible). We conclude that  $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$  is log canonical.

Now we consider the case where exactly one among  $a_2$  and  $b_2$  is zero. It is easy to check that  $B_P^\bullet$  is automatically smooth and a similar description to the one above applies, with the only difference that the restriction  $(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}$  can acquire a triple intersection point. But then we can apply Lemma 5.6 to conclude that  $\left(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet}\right)$  is log canonical.

For the ampleness condition, on  $B_P^\bullet \cong \mathbb{F}_1$  denote by  $h$  a section of self-intersection 1 and by  $f$  a fiber. Then

$$K_{B_P^\bullet} + D_P^\bullet|_{B_P^\bullet} + \left(\frac{1+\epsilon}{2}\right)(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet} \sim -2h - f + h + \left(\frac{1+\epsilon}{2}\right)(2h + 2f) = \epsilon(h + f),$$

which is ample.  $\square$

To deal with the last case (type (d)), we need an additional preliminary lemma.

**Lemma 10.7.** *Let  $H_1, H_2, H_3$ , and  $H$  be distinct planes in  $\mathbb{A}^3$  through the origin in general position. Then  $(\mathbb{A}^3, \left(\frac{1+\epsilon}{2}\right)(H_1 + H_2 + H_3) + H)$  is log canonical.*

*Proof.* Consider the blow up morphism  $f: \text{Bl}_{(0,0,0)} \mathbb{A}^3 \rightarrow \mathbb{A}^3$  and let  $E$  be the exceptional divisor. If  $\alpha = \frac{1+\epsilon}{2}$ , then we have that

$$\begin{aligned} K_{\text{Bl}_{(0,0,0)} \mathbb{A}^3} + f_*^{-1}(\alpha(H_1 + H_2 + H_3) + H) &= f^*(K_{\mathbb{A}^3} + \alpha(H_1 + H_2 + H_3) + H) + aE \Rightarrow \\ 2E &= (3\alpha + 1)E + aE \Rightarrow \\ a &= (-3\alpha + 1) > -1. \end{aligned}$$

This implies that  $(\mathbb{A}^3, (\frac{1+\epsilon}{2})(H_1 + H_2 + H_3) + H)$  is log canonical because  $f$  is a log resolution of singularities.  $\square$

**Proposition 10.8.** *Given  $P$  of type  $(d)$ , then the pair  $(B_P^\bullet, D_P^\bullet|_{B_P^\bullet} + (\frac{1+\epsilon}{2})(\Delta_P^\bullet - D_P^\bullet)|_{B_P^\bullet})$  is stable (see Definition 5.4 for the definition of type  $(d)$ ).*

*Proof.*

$$\begin{aligned} X_P^\bullet &= (\mathbb{P}^1)^3 \ni ([X_0 : X_1], [Y_0 : Y_1], [Z_0 : Z_1]), \\ D_P^\bullet &= \emptyset \text{ (there is no conductor divisor in this case),} \\ \Delta_P^\bullet &= V(X_0 X_1 Y_0 Y_1 Z_0 Z_1), \\ B_P^\bullet &= V\left(\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k\right), \end{aligned}$$

where any two coefficients  $c_{ijk}$  and  $c_{i'j'k'}$  cannot be simultaneously zero if  $(i, j, k)$  and  $(i', j', k')$  are vertices of the same edge of the cube (this is because inside  $Q$  we cannot fit two corner cuts with apices lying on the same edge).

Let us first assume that  $B_P^\bullet$  is smooth (hence irreducible), which happens for a generic choice of the coefficients  $c_{ijk}$  by Bertini's Theorem (see [Har, Chapter II, Theorem 8.18]). Then the anticanonical class  $-K_{B_P^\bullet} = -(K_{(\mathbb{P}^1)^3} + B_P^\bullet)|_{B_P^\bullet} = (1, 1, 1)|_{B_P^\bullet}$  is ample and  $K_{B_P^\bullet}^2 = 6$ , implying that  $B_P^\bullet \cong \text{Bl}_3 \mathbb{P}^2$  (see [B, Exercise V.21(1)]). If all the  $c_{ijk}$  are nonzero, then the restriction  $\Delta_P^\bullet|_{B_P^\bullet}$  can be as in Proposition 3.14 (generic case), or some of these lines can break into the union of two incident  $(-1)$ -curves. If some coefficients  $c_{ijk}$  are zero, then the lines configuration  $\Delta_P^\bullet|_{B_P^\bullet}$  acquires triple intersection points. In any case, the pair  $(B_P^\bullet, (\frac{1+\epsilon}{2})\Delta_P^\bullet|_{B_P^\bullet})$  is log canonical (here we use Lemma 5.6 in case of triple intersection points).

Now assume that  $B_P^\bullet$  is irreducible but singular. Let  $p \in B_P^\bullet$  be a singular point. We prove that  $p$  is a singularity of type  $A_1$  and the only singular point of  $B_P^\bullet$ . Up to obvious symmetries, we can assume that  $p$  is in the form  $([1 : a], [1 : b], [1 : c])$ . Consider the invertible change of coordinates  $X'_0 = X_0$ ,  $X'_1 = X_1 - aX_0$  and so on. This automorphism sends  $B_P^\bullet$  to an isomorphic surface  $B'_P$  which is singular at  $p' = ([1 : 0], [1 : 0], [1 : 0])$ . If we set  $x' = \frac{X'_1}{X'_0}$ ,  $y' = \frac{Y'_1}{Y'_0}$  and  $z' = \frac{Z'_1}{Z'_0}$ , then the equation of  $B'_P$  in this affine patch is in the form

$$c_0 x' y' + c_1 x' z' + c_2 y' z' + c_3 x' y' z' = 0. \quad (1)$$

The coefficients  $c_0, c_1$  and  $c_2$  are nonzero because  $B'_P$  is irreducible. Therefore it is clear that the singularity is of type  $A_1$ . Now we show that  $p'$  is the only singularity of  $B'_P$ , whose equation is given by

$$c_0 X_1 Y_1 Z_0 + c_1 X_1 Y_0 Z_1 + c_2 X_0 Y_1 Z_1 + c_3 X_1 Y_1 Z_1 = 0, \quad (2)$$

where we intentionally dropped the “prime” signs for simplicity of notation. It is enough to check the affine patches  $X_0 Y_0 Z_0 \neq 0$ ,  $X_0 Y_0 Z_1 \neq 0$ ,  $X_0 Y_1 Z_1 \neq 0$  and  $X_1 Y_1 Z_1 \neq 0$  because of the symmetries of our equation.

- In  $X_0Y_0Z_0 \neq 0$  we have

$$\begin{cases} c_0xy + c_1xz + c_2yz + c_3xyz = 0, \\ c_0y + c_1z + c_3yz = 0, \\ c_0x + c_2z + c_3xz = 0, \\ c_1x + c_2y + c_3xy = 0. \end{cases}$$

Let us show that this system has no solution other than  $(0, 0, 0)$ . Observe that  $x = 0, y = 0$  or  $z = 0$  implies that  $x = y = z = 0$ . Therefore, assume  $x, y$  and  $z$  nonzero. But if we multiply the last equation of the system by  $z$ , then the first equation implies that  $c_0xy = 0$ , which is a contradiction.

- In  $X_0Y_0Z_1 \neq 0$  we have to study the solutions of

$$\begin{cases} c_0xyz + c_1x + c_2y + c_3xy = 0, \\ c_0yz + c_1 + c_3y = 0, \\ c_0xz + c_2 + c_3x = 0, \\ c_0xy = 0. \end{cases}$$

But this system has no solutions because  $x = 0$  or  $y = 0$  implies  $c_2 = 0$  or  $c_1 = 0$  respectively.

- In  $X_0Y_1Z_1 \neq 0$  consider

$$\begin{cases} c_0xz + c_1xy + c_2 + c_3x = 0, \\ c_0z + c_1y + c_3 = 0, \\ c_1x = 0, \\ c_0x = 0, \end{cases}$$

which has no solutions because  $x = 0$  implies  $c_2 = 0$ .

- In the affine patch  $X_1Y_1Z_1 \neq 0$ , we obviously have no singular points because the dehomogenization of equation (2) is linear.

It is easy to see that the singular point  $p$  can lie on at most one irreducible component of  $\Delta_P^\bullet|_{B_P^\bullet}$ . To prove that  $(B_P^\bullet, (\frac{1+\epsilon}{2}) \Delta_P^\bullet|_{B_P^\bullet})$  is log canonical, we show that  $(X_P^\bullet, (\frac{1+\epsilon}{2}) \Delta_P^\bullet + B_P^\bullet)$  is log canonical and then we use inversion of adjunction (see [Ka]). This is done in two steps.

- $(X_P^\bullet, (\frac{1+\epsilon}{2}) \Delta_P^\bullet + B_P^\bullet)$  is log canonical in a neighborhood of a quadruple intersection point  $p$  of  $\Delta_P^\bullet + B_P^\bullet$ . Assume without loss of generality that  $p = ([1 : 0], [1 : 0], [1 : 0])$ , and therefore  $c_{000} = 0$ . In an affine neighborhood of  $p$  the equation of  $B_P^\bullet$  becomes

$$c_{100}x + c_{010}y + c_{001}z + c_{110}xy + c_{101}xz + c_{011}yz + c_{111}xyz = 0.$$

where we must have  $c_{100}, c_{010}, c_{001}$  nonzero. The affine equations for  $\Delta_P^\bullet$  at  $p$  are  $x = 0, y = 0, z = 0$ . Therefore, locally at  $p$ , the four irreducible components of  $\Delta_P^\bullet + B_P^\bullet$  are in general linear position. Now we use Lemma 10.7;

- $(X_P^\bullet, (\frac{1+\epsilon}{2}) \Delta_P^\bullet + B_P^\bullet)$  is log canonical in the complement of the quadruple intersection points. This is true in the complement of  $\Delta_P^\bullet$  because  $B_P^\bullet$  has at most an  $A_1$  singularity. Let  $H \subset \Delta_P^\bullet$  be an irreducible component. We show that  $(X_P^\bullet, \Delta_P^\bullet + B_P^\bullet)$  is log canonical in a neighborhood of  $H$  away from the quadruple intersection points. But this follows from inversion of adjunction because it is easy to observe that  $(H, (\Delta_P^\bullet - H + B_P^\bullet)|_H)$  is log canonical away from the quadruple intersection points (more precisely,  $H \cong \mathbb{P}^1 \times \mathbb{P}^1$ ,  $(\Delta_P^\bullet - H)|_H$  gives the toric boundary, and  $B_P^\bullet|_H$  is a  $(1, 1)$ -curve with no components in common with the toric boundary).

We are left with the case  $B_P^\bullet$  reducible. Decompositions into two irreducible components are given by

$$(aX_0Y_0 + bX_0Y_1 + cX_1Y_0 + dX_1Y_1)(eZ_0 + fZ_1),$$

for any choice of nonzero coefficients such that  $ad \neq bc$ . It is easy to see that these two components are isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  and how they are glued together. Decompositions into three irreducible components are given by

$$(aX_0 + bX_1)(cY_0 + dY_1)(eZ_0 + fZ_1),$$

for any choice of nonzero coefficients. Up to  $\text{Aut}((\mathbb{P}^1)^3)$  there is only one choice of coefficients. In both cases  $(B_P^\bullet, (\frac{1+\epsilon}{2})\Delta_P^\bullet|_{B_P^\bullet})$  is semi log canonical.

To conclude,  $K_{B_P^\bullet} + (\frac{1+\epsilon}{2})\Delta_P^\bullet|_{B_P^\bullet}$  is ample because it is the pullback to  $B_P^\bullet$  of the following divisor:

$$K_{X_P^\bullet} + B_P^\bullet + \left(\frac{1+\epsilon}{2}\right)\Delta_P^\bullet \sim \epsilon(1, 1, 1),$$

which is ample. □

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